TUT WURI HANDAYANI

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FOREWORDS

President of the Indonesian Mathematical Society (IndoMS)

FOREWORDS

Assalamu'alaikum Warahmatullahi Wabarakatuh

Good morning and best wishes for all of us

It is my pleasure to say that the proceedings of the Second IndoMS International Conference on Mathematics and Its Applications (IICMA) 2013 from November 6 to November 7 at Yogyakarta-Indonesia finally published. The IICMA 2013 is the second IICMA, after IICMA 2009, which is organized by the Indonesian Mathematical Society (IndoMS) in collaboration with Department of Mathematics, Faculty of Mathematics and Natural Sciences, Gadjah Mada University and funded by Directorate of Research and Community Services, the Directorate General of Higher Education, Ministry of Education and CultureRepublic of Indonesia.

IICMA 2013 is one of the activities of IndoMS period 2012-2014. Organizing an IICMA 2013 is not only a continuing academic activity for IndoMS, but it is also a good opportunity for discussion, dissemination of the research result on mathematics including: Analysis, Applied Mathematics, Algebra, Theoretical Computer Science, Mathematics Education, Mathematics of Finance, Statistics and Probability, Graph and Combinatorics, also to promote IndoMS as a non-profit organization which has a member more than 1,400 people from around Indonesia area.

We would like to express our sincere gratitude to all of the Invited Speakers from the Netherlands, Georgia, India, Germany, Singapore and also Indonesia from Universities (ITB, UPI, University of Jember) and LAPAN Bandung, all of the speakers, members and staffs of the organizing committee of IICMA 2013. Special thanks to the Secretary of International Mathematics Union (IMU), the Directorate General of Higher Education, the Dean of Faculty of Mathematics and Natural Sciences-Gadjah Mada University, the Head of Department of Mathematics together with all staffs and students, also for supporting of lecturers and staffs as an organizing committee from Indonesian University, Padjadjaran University, University North of Sumatera, Sriwijaya University and Bina Nusantara University. Finally, we also would like to to give a big thanks for all reviewers who help us to review all papers which are submitted after IICMA.

With warmest regards,

Budi Nurani Ruchjana
President IndoMS 2012-2014

Chair of the Committee IICMA 2013

On behalf of the Organizing Committee of IndoMS International Conference on Mathematics and its Applications (IICMA) 2013, I would like to thanks all participants of the conference. This conference was organized by Indonesia Mathematical Society (IndoMS) and hosted by Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Gadjah Mada, Yogyakarta, Indonesia, during 6-7 November 2013.

In IICMA 2013, there will be 122 talks which consists of 10 invited and 112 contributed talks coming from diverse aspects of mathematics ranging from Analysis, Applied Mathematics, Algebra, Theoretical Computer Science, Mathematics Education, Mathematics of Finance, Statistics and Probability, Graph and Combinatorics. However, the number of paper which were sent and accepted in this proceedings is 33 papers. We would also like to give our

- Prof. Dr. S. Arumugam (Combinatorics, Kalasalingam University, India)
- Prof. Dr. Bas Edixhoven (Algebra, Universiteit Leiden-the Netherlands)
- Prof. Dr. Dr. h.c. mult. Martin Grotschel (Applied Math, Technische Universitat Berlin, Germany and International Mathematics Union)
- Prof. Dr. Kartlos Joseph Kachiashvili (Statistics, Tbilisi State University-Georgia)
- Prof. Dr. Berinderjeet Kaur (Mathematics Education, National Institute of Education, Singapore)
- Prof. Hendra Gunawan, Ph.D (Analysis, ITB-Bandung, Indonesia)
- Prof. Dr. Edy Hermawan (Atmospheric Modeling, LAPAN Bandung)
- Prof. H. Yaya S. Kusumah, M.Sc., Ph.D (Mathematics Education, UPI-Bandung, Indonesia)
- Prof. Dr. Slamin (Combinatorics, Universitas Jember, Indonesia)
- Dr. Aleams Barra (Algebra, ITB-Bandung, Indonesia)

We thank all who sent the papers or proceedings of IICMA 2013. We also would like to give our gratitude for all reviewers who worked hard for making this proceedings done.

IndoMS conveys high appreciation for the Directorate General of Higher Education (DGHE) for the most valuable support in organizing the conference. We also would like to give our gratitude to Universitas Gadjah Mada, especially to Department of Mathematics, Faculty of Mathematics and Natural Sciences for providing the places and staffs for this conference.

It remains to thank all members of Organizing Committee spread across 3 cities, Depok, Bandung and Yogyakarta who have worked very hard to make this conference happens.

Yogyakarta, January 5th, 2014
On behalf of the Committee
Dr. Kiki Ariyanti Sugeng - Chair.

ACKNOWLEDGEMENT

The organizing Committee of the IICMA 2013 and Indonesian Mathematical Society (IndoMS) wish to express their gratitude and appreciation to all Sponsors and Donors for their help and support for the Program, either in form of financial support, facilities, or in other form. The Committee addresses great thank especially to:

a. Directorate General of Higher Education (DGHE)
b. The Rector of the Gadjah Mada University
c. The Dean of Faculty ofMathematics and Natural Sciences, Gadjah Mada University.
d. The Head of the Department of Mathematics, Gadjah Mada University.
e. All sponsors of the conference

The Committee extends its gratitude to all invited speakers, parallel session speakers, and guests for having kindly and cordially accepted the invitation and to all participants for their enthusiastic response.

Finally the Committee also would like to acknowledge and appreciate for the support and help of all IndoMS members in the preparation for and the running of the program

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INVITED SPEAKERS

PROTECTION OF A GRAPH

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Abstract. Let $G = (V, E)$ be a graph and let $k \geq 0$ be an integer. A function $f : V \to \{0, 1, 2, \ldots, k\}$ is called a safe function if every vertex v with $f(v) = 0$ isadjacent to at least one vertex u with $f(u) > 0$. A vertex v with $f(v) = 0$ issaid to be defended by a vertex u with $f(u) > 0$, if $uv \in E$ and if the function$f_{vu} : V \to \{0, 1, \ldots, k\}$ defined by $f_{vu}(u) = f(u) - 1, f_{vu}(v) = 1$ and $f_{vu}(x) = f(x)$for all $x \in V - \{u, v\}$, is again a safe function.In this talk we present several graph theoretic parameters based on safe functions.

Key words and Phrases: adjacent, defended, safe function.

References

[1] J. Alber, H. Fernau, and R. Niedermeier, Parametrized complexity: Exponentialspeed-up for planar graph problems, J. Algorithms, 52 (2004), 26–56.

[2] S. Arumugam, Karam Ebadi and Mart´ın Manrique, Co-secure domination ingraphs, Util. Math., (accepted).

[3] K. S. Booth, Dominating sets in chordal graphs, Research Report CS-80-34,Univ. of Waterloo, 1980.

[4] K. S. Booth and J. H. Johnson, Dominating sets in chordal graphs, SIAM J.Comput., 11 (1982), 191–199.

[5] A. P. Burger, E. J. Cockayne, W. R. Grundlingh, C. M. Mynhardt, J. H. VanVuuren, and W. Winterbach, Finite order domination in graphs, J. Combin.Math. Combin. Comput., 49 (2004), 159–175.

[6] A. P. Burger, E. J. Cockayne, W. R. Grundlingh, C. M. Mynhardt, J. H. VanVuuren, and W. Winterbach, Infinite order domination in graphs, J. Combin.Math. Combin. Comput., 50 (2004), 179–194.

[7] A.P. Burger, Michael A. Henning, and Jan H. van Vuuren, Vertex covers andsecure domination in graphs, Quaest. Math., 31 (2008), 163–171.

[8] E. J. Cockayne, P. A. Dreyer Jr., S. M. Hedetniemi and S. T. Hedetniemi,Roman domination in graphs, Discrete Math., 278 (2004), 11–22.

[9] E. J. Cockayne, Irredundance, secure domination and maximum degree intrees, Discrete Math., 307 (2007), 12–17.

[10] E. J. Cockayne, O. Favaron, and C. M. Mynhardt, Secure domination, weakRoman domination and forbidden subgraphs, Bull. Inst. Combin. Appl., 39(2003), 87–100.

[11] E. J. Cockayne, P. J. P. Grobler, W. R. Grundlingh, J. Munganga, and J.H. van Vuuren, Protection of a graph, Util. Math., 67 (2005), 19–32.

[12] G. Gunther, B. L. Hartnell, L. Markus, and D. F. Rall, Graphs with uniqueminimum dominating sets, Congr. Numer. ,101 (1994), 55–63.

[13] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater (eds), Fundamentals ofDomination in Graphs, Marcel Dekker, Inc. New York, 1998.

[14] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater (eds), Domination inGraphs: Advanced Topics, Marcel Dekker, Inc. New York, 1998.

[15] M. A. Henning and S. M. Hedetniemi, Defending the Roman Empire-A newstrategy, Discrete Math., 266 (2003), 239–251.

[16] C. M. Mynhardt, H. C. Swart, and E. Ungerer, Excellent trees and securedomination, Util. Math., 67 (2005), 255–267.

COUNTING QUICKLY THE INTEGER VECTORS WITH A GIVEN LENGTH

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Abstract. The question is how one can compute the number of ways in which an integer m can be written as a sum of n squares of integers, fast. There are explicit formulas for these numbers for n even up to 10, due to Fermat, Legendre, Gauss, Jacobi, Eisenstein, Smith and Liouville. I will explain that for even n greater than 10 there are no such formulas anymore, but that one can still compute these numbers, for n even and m given with its factorisation in prime numbers, in a running time at most a power of n.log(m) (assuming the Riemann hypothesis for number fields). This is an application of a generalisation by Peter Bruin of joint work of the speaker with Jean-Marc Couveignes, Robin de Jong and Franz Merkl.

Key words and Phrases: F modular forms, Theta Function, Identities, algorithms, Galois symmetry.

References

[1] Computational aspects of modular forms and Galois representations. Edited with J-M. Couveignes, and with contributions by Johan Bosman, Jean-Marc Couveignes, Bas Edixhoven, Robin de Jong, and Franz Merkl. Volume 176 of "Annals of Mathematics Studies", Princeton University Press, 2011. Also free on-line: http://www.math.u-bordeaux1.fr/~jcouveig/book.htm

[2] Edixhoven, B and Couveignes, J.M, 2012, Approximate computations with modular curves. arXiv:1205.5896v1 [math.NT]

[3] P. Bruin. 2010, Modular curves, Arakelov theory, algorithmic applications. PhD thesis,Leiden, http://www.math.leidenuniv.nl/nl/theses/220/.

INVESTIGATION OF CONSTRAINED BAYESIAN METHODS OF HYPOTHESES TESTING WITH RESPECT TO CLASSICAL METODS

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Abstract. The article focuses on the discussion of basic approaches to hypotheses testing, which are Fisher, Jeffreys, Neyman, Berger approaches and a new one proposed by the author of this paper and called the constrained Bayesian method (CBM). Wald and Berger sequential tests and the test based on CBM are presented also. The positive and negative aspects of these approaches are considered on the basis of computed examples. Namely, it is shown that CBM has all positive characteristics of the above-listed methods. It is a data-dependent measure like Fisher's test for making a decision, uses a posteriori probabilities like the Jeffreys test and computes error probabilities Type I and Type II like the Neyman-Pearson's approach does. Combination of these properties assigns new properties to the decision regions of the offered method. In CBM the observation space contains regions for making the decision and regions for no-making the decision. The regions for no-making the decision are separated into the regions of impossibility of making a decision and the regions of impossibilityof making a unique decision. These properties bring the statistical hypotheses testing rule in CBM much closer to the everyday decision-making rule when, at shortage of necessary information, the acceptance of one of made suppositions is not compulsory. Computed practical examples clearly demonstrate high quality and reliability of CBM. In critical situations, when other tests give opposite decisions, it gives the most logical decision. Moreover, for any information on the basis of which the decision is made, the set of error probabilities is defined for which the decision with given reliability is possible.

Key words and Phrases: hypotheses testing, likelihood ratio, frequentist approaches, Bayesian approach, constrained Bayesian method.

References

[1] Bauer P. at all., 1988, *Multiple Hypothesenprüfung*. (Multiple Hypotheses Testing.) Berlin: Springer-Verlag (In German and English).

[2] Berger, J.O., 2003, Could Fisher, Jeffreys and Neyman have Agreed on Testing? *Statistical Science*, 18, 1–32.

[3] Berger J.0., Boukai B. and Wang Y., 1997, Unified Frequentist and Bayesian Testing of a Precise Hypothesis, *Statistical Science*, 12, 3, 133-160.

[4] Berger, J.O., Boukai, B. and Wang, Y., 1999, Simultaneous Bayesian–frequentist sequential testing of nested hypotheses. *Biometrika*,86,79–92.

[5] Berger, J.O., Brown, L.D. and Wolpert, R.L., 1994, A Unified Conditional Frequentist and Bayesian Test for Fixed and Sequential Simple Hypothesis Testing. *The Annals of Statistics*, 22, 4, 1787-1807.

[6] Berger, J.O. and Delampady, M., 1987, Testing precise hypothesis (with discussion). *Statist. Sci*. 2 317-352.

[7] Berger, J.O. and Sellke, T., 1987, Testing a point null hypothesis. The irreconcilability of p-values and evidence (with discussion). *J. Amer. Statist. Assoc*. 82 112-139.

[8] Berger, J.O. and Wolpert, R.L., 1984, *The Likelihood Principle*. IMS, Hayward, CA.

[9] Berger, J.O. and Wolpert, R.L., 1988, *The Likelihood Principle*, 2nd ed. (with discussion). IMS, Hayward, CA.

[10] Berk, R.H., 1966, Limiting behavior of posterior distributions when the model is incorrect. Ann. Math. Statist. 37, 51-58.

[11] Bernardo, J.M., 1980, A Bayesian analysis of classical hypothesis testing, Universidad de Valencia, 605-617.

[12] Bernardo,J.M.and Rueda, R., 2002, Bayesian Hypothesis Testing: A Reference Approach.*International Statistical Review,* p. 1-22.

[13] Braun, H.I., 1994, *The Collected Works of John W. Tukey*. Vol. VIII: *Multiple Comparisons: 1948-1983*. New York: Chapman & Hall.

[14] Brownie, C. and Keifer, J., 1977, The ideas of conditional con- fidence in the simplest setting. Comm. Statist. Theory Meth- ods 6 691-751.

[15] Cassela, G. and Berger, R.L., 1987, Reconciling evidence in the one-sided testing problem (with discussion). *J. Amer. Statist. Assoc*. 82 106-112.

[16] Casella, G. and Wells, M.T., 1990, Comparing p-values to Neyman-Pearson tests. Technical Report BU-1073-M, Biometrics Unit and Statistics Cent., Cornell Univ.

[17] Christensen, R., 2005, Testing Fisher, Neyman, Pearson, and Bayes,*The American Statistician, May 2005, Vol. 59, No. 2, 121-126*.

[18] Dass, S.C. and Berger, J.O., 2003, Unified Conditional Frequentist and Bayesian Testing of Composite Hypotheses. *Scandinavian Journal of Statistics*, Vol. 30, No. 1, pp. 193-210.

[19] Delampady, M. and Berger, J.0., 1990, Lower bounds on Bayes factors for the multinomial distribution, with application to chi-squared tests of fit. *Ann. Statist.*, 18, 1295-1316

[20] Edwards, W., Lindman, H. and Savage, L.J., 1963, Bayesian statistical inference for psychological research. *Psychological Review*, 70, 193-242

[21] Efron, B., 2004, Large-Scale Simultaneous Hypothesis Testing. *Journal of the American Statistical Association*. Vol. 99, Issue 465, pages 96-104.

[22] Fisher, R.A., 1925, Statistical Methods for Research Workers: London: Oliver and Boyd.

[23] Gómez-Villegas, M.A., and González-Pérez, B., 2011, A Bayesian Analysis for the Homogeneity Testing Problem Using ε–Contaminated Priors. *Communications in Statistics - Theory and Methods*. Vol. 40, Issue 6, pages 1049-1062.

[24] Gómez–Villegas, M.A., Maín, P. and Sanz, L., 2009, A Bayesian analysis for the multivariate point null testing problem. *Statistics*. Vol. 43, Issue 4, pages 379-391.

[25] Good, I. J., 1992, The Bayesian/Non-Bayesian compromise: a brief review. J Amer. Statist. Assoc. 87 597-606.

[26] Hochberg, Y. and Tamhance, A.C., 1987, *Multiple Comparison Procedures*. New York: Wiley.

[27] Hoppe, F.M., 1993, *Multiple Comparisons. Selection, and Applications in Biometry*. New York: Dekker.

[28] Hsu, J.C., 1996, *Multiple Comparisons: Theory and methods*. New York: Chapman & Hall.

[29] Hubbard, R. and Bayarri, M.J., 2003, Confusion Over Measures of Evidence (p's) Versus Errors (α's) in Classical Statistical Testing, *The American Statistician*, 57, 171–177.

[30] Hwang, J.T., Casella, G., Robert, Ch., Wells, M.T. and Farrell, R.H., 1992, Estimation of Accuracy in Testing, *The Annals of Statistics*, 20, 1, 490-509.

[31] Jeffreys, H., 1939, Theory of Probability. 1st ed. Oxford: The Clarendon Press.

[32] Kachiashvili, K.J., 1989, *Bayesian algorithms of many hypothesis testing*. Ganatleba, Tbilisi, 144 p.

[33] Kachiashvili, K.J., 2003, Generalization of Bayesian Rule of Many Simple Hypotheses Testing. *International Journal of Information Technology & Decision Making,* World Scientific Publishing Company, 2, 1, 41-70.

[34] Kachiashvili, K.J., 2011, Investigation and Computation of Unconditional and Conditional Bayesian Problems of Hypothesis Testing. *ARPN Journal of Systems and Software*, Vol.1 No.2, May 2011, 47-59.

[35] Kachiashvili, K.J., 2013, The Methods of Sequential Analysis of Bayesian Type for the Multiple Testing Problem. *Sequential Analysis*, 4: -. (It is accepted for publication)

[36] Kachiashvili, G.K., Kachiashvili, K.J. and Mueed, A., 2012, Specific Features of Regions of Acceptance of Hypotheses in Conditional Bayesian Problems of Statistical Hypotheses Testing. *Sankhya, Series A*, Vol. 74 A1. The original publication is available at sankhya.isical.ac.in.

[37] Kachiashvili, K.J. and Hashmi, M.A., 2010, About Using Sequential Analysis Approach for Testing Many Hypotheses. *Bulletin of the Georgian Academy of Sciences*, vol. 4, no. 2, 20-25.

[38] Kachiashvili, K.J., Hashmi, M. A. and Mueed, A., 2012, Sensitivity Analysis of Classical and Conditional Bayesian Problems of Many Hypotheses Testing. *Communications in Statistics—Theory and Methods*, Volume 41, Issue 4, 591–605, 2012.

[39] Kachiashvili, K.J. and Mueed, A., 2011, Conditional Bayesian Task of Testing Many Hypotheses, *Statistics*, DOI:10.1080/02331888.2011.602681. http://dx.doi.org/10.1080/02331888.2011.602681

[40] Klockars, A.J. and Sax, G., 1986, Multiple Comparison. Newbury Park, CA: Sage.

[41] Kiefer, J., 1977, Conditional confidence statement and confidence estimations (with discussion). *J. Amer. Statist. Assoc.* 72, 360, 789-808.

[42] Lehmann, E.L., 1993, The Fisher, Neyman-Pearson Theories of Testing Hypotheses: One Theory or Two? *American Statistical Association Journal* Theory and Methods, 88, 424, 1242-1249.

[43] Lehmann, E. L., 1997, *Testing Statistical Hypotheses* (2nd ed.), New York: Springer.

[44] Miller, R.G., 1966, *Simultaneous Statistical Inference*. New York: Wiley.

[45] Moreno, E. and Giron, F.J., 2006, On the frequentist and Bayesian approaches to hypothesis testing, SORT 30 (1) January-June 2006, 3-28.

[46] Neyman, J. and Pearson, E., 1928, On the use and Interpretation of Certain Test Criteria for Purposes of Statistical Inference. Part I, *Biometrica*, 20A, 175-240.

[47] Neyman, J. and Pearson, E., 1933, On the Problem of the Most Efficient Tests of Statistical Hypotheses, *Philos. Trans. Roy. Soc., Ser. A*, 231, 289-337.

[48] Rao, C.R., 2006, *Linear Statistical Inference and Its Application*. Second Edition. Wiley, New York.

[49] Rosenthal, R. and Rubin, D.B., 1983, Psychological Bulletin, Vol 94(3), 540-541.

[50] Sage, A.P. and Melse, J.L., 1972, *Estimation Theory with Application to Communication and Control*. McGraw-Hill, New York.

[51] Schaarfsma, W., Tobloom, J. and Van der Menlen, B., 1989, Discussing truth or falsity by computing a q-value. In *Statistical Data Analysis and Inference* (Y. Dodge, ed.) 85-100. North-Holland, Amsterdam.

[52] Shaffer, J.P., 1995, Multiple hypothesis testing, *Annual Review of Psychology* (*Annu. Rev. Psychol.*), 46: 561-84.

[53] Shaffer, J.P., 1986, Modified Sequentially Rejective Multiple Procedures. *Journal of the American Statistical Association*. Vol. 81, No. 395, pp. 826-831.

[54] Toothaker, L.E., 1991, *Multiple Comparisons for Researchers*. NewBury Park, CA: Sage.

[55] Wald, A., 1947a, *Sequential analysis*. Wiley, NY.

[56] Wald, A., 1947b, Foundations of a General Theory of Sequential Decision Functions. *Econometrica*, 15, 279-313.

[57] Westfall, P.H., Johnson, W.O. and Utts, J.M., 1997, A Bayesian Perspective on the Bonferroni Adjustment. *Biometrica*, 84, 2, 419-427.

[58] Westfall, P.H. and Young, S.S., 1993, *Resampling-based Multiple Testing*. New York: Wiley.

[59] Wolpert, R.L., 1996, Testing simple hypotheses. In *Data Analysis and Information Systems* (H. H. Bock andW. Polasek, eds.) **7** 289–297. Springer, Heidelberg.

MATHEMATICS EDUCATION IN SINGAPORE – AN INSIDER'S PERSPECTIVE

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Abstract. Singapore's Education System has evolved over time and so has Mathematics Education in Singapore. The present day School Mathematics Curricula can best be described as one that caters for the needs of every child in school. It is based on a framework that has mathematical problem solving as its primary focus. The developments from 1946 to 2012 that have shaped the present School Mathematics Curricula in Singapore are direct consequences of developments in the Education System of Singapore during the same period. The curriculum, teachers, leaners and the learning environment may be said to contribute towards Singapore's performance in international benchmark studies such as TIMSS and PISA.

Key words and Phrases: mathematics education, Singapore, differentiated curriculum, TIMSS, PISA.

References

[1] Yip Yip, S.K.J., Eng, S.P. & Yap, Y.C.J. 1990. 25 Years of educational reform. In J.S.K. Yip & W.K. Sim (Eds.), *Evolution of educational excellence – 25 Years of education in the Republic of Singapore* (pp. 1-30). Singapore: Longman Singapore Publishers (Pte) Ltd.

[2] Ministry of Education. (1979). *Report on the Ministry of Education by Dr Goh and his team*. Singapore: Ministry of Education.

[3] Lee, K.Y. (1979). Letter in response to the report on the Ministry of Education by Dr Goh and his team. In *Ministry of Education, Report on the Ministry of Education by Dr Goh and his team*. Singapore: Ministry of Education.

[4] Economic Committee (1986). *Report of the Economic Committee The Singapore Economy: New Directions*. (Chaired by BG Lee Hsien Loong). Singapore: Ministry of Trade and Industry.

[5] Tan, T.K.Y. (1986). *Speech delivered at the Nanyang Technological Institute*, 22 July 1986.

[6] Ministry of Education. (1987). *Towards Excellence in Schools*. Singapore: Ministry of Education.

[7] Tan, T.K.Y. (1987). *Speech delivered at the First Schools Council Meeting*. Reported in Straits Times, 14 Jan 1987.

[8] Kaur, B. (2002). Singapore's school mathematics curriculum for the 21st century. In J. Abramsky, *Reasoning, explanation and proof in school mathematics and their place in the intended curriculum-Proceedings of the QCA International Seminar*, 4-6 October 2001, pp 166-177. London: Qualifications and Curriculum Authority, UK.

[9] Goh, C.T. (1997). Shaping our future: "Thinking Schools" and a "Learning Nation". *Speeches*, 21(3): 12-20. Singapore: Ministry of Information and the Arts.

[10] Ministry of Education (MOE). (1998). *Mathematics Newsletter*, 1(17). Singapore: Curriculum Planning and Development Division, Ministry of Education.

[11] Ministry of Education (MOE). 1997. *Towards Thinking Schools*. Singapore: Ministry of Education.

[12] Ministry of Education. (2000). *Proceedings of MOE work plan seminar: Ability-Driven Education – Making it Happen*. Singapore: Ministry of Education.

[13] Wong, K.Y. (1991). Curriculum development in Singapore. In C. Marsh, & P. Morris (Eds.), *Curriculum development in East Asia* (pp. 129-160). London: Falmer Press.

[14] Lee, P.Y. & Fan, L. (2002). *The development of Singapore mathematics curriculum – understanding the changes in syllabus, textbooks and approaches*. A talk given at the Chongqing conference 17 – 20 August 2002. Unpublished.

[15] Ministry of Education. (2006). *A guide to teaching and learning of O level mathematics 2007*. Singapore: Curriculum Planning & Development Division, Ministry of Education.

[16] Ministry of Education. (2006). *A guide to teaching and learning of N level (Academic) mathematics 2007*. Singapore: Curriculum Planning & Development Division, Ministry of Education.

[17] Ministry of Education. (2006). *A guide to teaching and learning of N level (Technical) mathematics 2007*. Singapore: Curriculum Planning & Development Division, Ministry of Education.

[18] Ministry of Education. (2006). *A guide to teaching and learning of primary mathematics 2007*. Singapore: Curriculum Planning & Development Division, Ministry of Education.

[19] Ministry of Education. (1992). *Mathematics syllabus-Secondary 1 and 2 Normal (Technical) Course*. Singapore: Curriculum Planning Division, Ministry of Education.

[20] Ministry of Education (2012). *O-level mathematics teaching and learning syllabus*. Singapore: Curriculum Planning & Development Division, Ministry of Education.

[21] Ministry of Education (2012). *N(A)-level mathematics teaching and learning syllabus*. Singapore: Curriculum Planning & Development Division, Ministry of Education.

[22] Ministry of Education (2012). *N(T)-level mathematics teaching and learning syllabus*. Singapore: Curriculum Planning & Development Division, Ministry of Education.

[23] Goh, C.T. (2001). *Shaping lives, moulding nation*. Speech at the Teachers' Day Rally, Friday 31st August 2001. Singapore: Ministry of Education.

[24] The National Institute of Education. (2002). *Moulding lives, shaping tomorrow – The NIE story*. Singapore: The National Institute of Education.

[25] Ministry of Education. (undated). *Enhanced Performance Management System*. Singapore: Ministry of Education.

[26] Mckinsey & Co. (2007). *How the world's best-performing school systems come out on top*. Mckinsey & Co Report.

MODELING, SIMULATION ANDOPTIMIZATION: EMPLOYING MATHEMATICSIN PRACTICE

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Abstract. Unlike engineering, physics, and chemistry, mathematics isalmost invisible in daily life; even many mathematiciansare unaware that significant applications of mathematicsare ubiquitous in our environment and activities.
In my lecture I will outline where and how mathematics isemployed in practice today. A first step is the mathematicalmodeling of real-world phenomena, engineering tasks,business processes, and the like. Using simulation toolsthe "correctness" of the models designed is checked,and afterwards optimization methods are employed to solvethe problems considered to optimality or to "practicalsatisfaction".I call this way of addressing real-world problems the"application driven approach" and view this as a major contribution of mathematics to the industrial creation of value. Mathematics has, in the last years, become a keyin the handling of complex systems and is playing an importantrole as a production factor and amplifier for innovations.
In my lecture I will substantiate my claims by providingnumerous examples of projects (of my own research group andmy colleagues in Berlin) with industry and partners from othersciences to demonstrate successes (and some failures) of thismethodology. A wide range of real applications will be coveredincluding problems in transport, traffic and logistics, energy,telecommunication, manufacturing, medicine and the life sciences.Mathematics alone does not suffice, though. Close and faithfulcooperation with engineers, management and computer scientists,as well as researchers and practitioners in various otherfields are fundamental for significantsuccess in practice.

Key words and Phrases: Applied mathematics, mathematics in industry and other sciences, modelling, simulation, optimization

STRONG AND WEAK TYPE INEQUALITIES FOR FRACTIONAL INTEGRAL OPERATORS ON GENERALIZED MORREY SPACES

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Abstract. Fractional integral operators were first studied by G.H. Hardy & J.E.Littlewood in the 1920's and by S.L. Sobolev in the 1930's. They proved the famous inequality for the operators on Lebesgue spaces, known as the Hardy-Littlewood-Sobolev inequality. Much development has been made since then, especially in the last two decades. In this talk, some recent results on the strong and weak type inequalities for the operators on generalized Morrey spaces (of non-homogeneous type) will be presented. Some related results and basic techniques to obtain the inequalities will be exposed.

Key words and Phrases: Fractional integral operators, strong and weak type inequalities, generalized Morrey spaces.

References

[1] Eridani; Gunawan, H.; Nakai, E. and Sawano, Y., 2013, Characterizations for the generalized fractional integral operators on Morrey spaces, *Mathematics Inequalities and Applications*, to appear.

[2] Gunawan, H.; Hakim, D.I.; Sawano, Y. and Sihwaningrum, I., 2013, Weak type inequalities for some singular integral operators on generalized non-homogeneous Morrey spaces, submitted.

THE ENDLESS LONG-TERM PROGRAMS OF TEACHER PROFESSIONAL DEVELOPMENT FOR ENHANCING STUDENT'S ACHIEVEMENT IN MATHEMATICS

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Abstract. To be proficient in teaching mathematics, mathematics teachers need to master mathematics, school mathematics, and mathematical pedagogy. They have to comprehend the knowledge of student understanding and thinking, since all these aspects will definitely guide teachers in constructing important mathematical tasks. To develop their professional skills, teachers should be given opportunities to attend some educational meetings where special and structured programs are designed; to be involved in collegial interactions, where they can develop and apply various methods and approaches; and to conduct reflection sessions based on their knowledge, learning experience, and best teaching practices. Basically, learning mathematics is an integrated process, and so is learning about mathematics teaching and long-term program for professional development. In this seminar, some activities related to mathematics teachers professional development and some findings based on observations during professional development programs on mathematics will be analyzed and discussed.

Key words and Phrases: Mathematics teacher professional development, teaching and learning process, collegial interactions, teaching proficiency.

DIGRAPH CONSTRUCTION TECHNIQUES AND THEIR CLASSIFICATIONS

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Abstract. A communication network can be modelled as a *graph*or a *directed graph* (*digraph*), where each processing element is represented by a vertex and the connection between two processing elements is represented by an *edge* (or, in case of directed connections, by an *arc*). The problem of constructing large communication network, subject to the constraints that the number of connections which can be attached to a processing element is limited and that a short communication route between any two processing elements is required, corresponds to the well-known fundamental problem called the *degree/ diameter problem*: construct graphs with the largest possible number of vertices (order) for given degree and diameter. The directed version of the problems differs only in that `degree' is replaced by `out-degree' in the statement of the problems. There are two mainstreams of research activities related to the {\it degree/diameter problem}, namely, (a) proving the non-existence of graphs or digraphs of order `close' to the Moore bound and so lowering the upper bound on the order; and (b) constructing large graphs or digraphs and so incidentally obtaining better lower bounds on the order.

There are many ways to construct a graph or a digraph, for example, drawing by hand, using computer search, using an algebraic specification and making use of existing graphs or digraphs to obtain new graphs using some particular construction technique. In this presentation, we classify the construction techniques of large directed graphs. We first presentthe construction techniques that have been established, namely, the generalised de Bruijn digraphs,generalised Kautz digraphs, line digraphs, digon reduction, generalised digraphs on alphabets, partialline digraphs, digraphs constructed by the use of voltage assignments, and vertex deletion scheme as well as a new construction technique using adjacency matrix. Then we classify them according to the general method of generating newdigraphs; the diregularity of generated digraphs; and the range of orders of the generated digraphs.

Key words and Phrases: construction technique, directed graph.

References

[1] Baskoro, E.T., Brankovic, L., Miller, M., Plesnık, J., Ryan, J. and Siran, J., 1997, Large digraphs with small diameter: A voltage assignment approach, JCMCC 24 (1997) 161–176.

[2] Baskoro E.T. & Miller, M.. 1996, A procedure for constructing a minimum diameter digraph, *Bulletin of ICA* 17 (1996) 8–14.

[3] Bridges, W.G. and Toueg, S., 1980. On the impossibility of directed Moore graphs, *Journal ofCombinatorial Theory* Series B 29 (1980) 339-341.

[4] Comellas, F., and Fiol, M.A., 1990, Using simulated annealing to design interconnection networks, *Technical report* DMAT-05-0290, Universitat Politecnica de Catalunya. Presented at the *Fifth SIAM Conference on Discrete Mathematics*, Atlanta (1990).

[5] Fiol, M.A., Yebra, J.L.A. and Alegre, I.,1984, Line digraph iterations and the (d,k) problem for directed Graphs, *IEEE Transactions on Computers* C-33 (1984), 400–403.

[6] Fiol, M.A., Llad´o, A.S. and Villar, J.L., 1988, Digraphs on alphabets and the (d,N) digraph problem, *Ars Combinatoria* 25C (1988) 105–122.

[7] Fiol, M.A. and Llado, A.S., 1992, The partial line digraph technique in the design of large interconnection networks, *IEEE Transactions on Computers* 41 (1992.) 848–857.

[8] Hoffman, A.J. and Singleton, R., 1960, On Moore graphs with diameter 2 and 3, *IBM J. Res. Develop.* 4, (1960) 497 – 504.

[9] Imase, M. and Itoh, M., 1981, Design to minimize diameter on building-block network, *IEEE Trans. on Computers* C-30 (1981) 439–442.

[10] Imase, M. and Itoh, M., 1983, A design for directed graphs with minimum diameter, *IEEE Trans. on Computers* C-32 (1983) 782–784.

[11] Miller, M. and Fris, I., 1988, Minimum diameter of diregular digraphs of degree 2,*Computer Journal* 31 (1988) 71–75.

[12] Miller, M., and Fris, I., 1992, Maximum order digraphs for diameter 2 or degree 2, *Pullman volume of Graphs dan Matrices, Lecture Notes in Pure dan Applied Mathematics* 139 (1992) 269-278.

[13] Miller, M. and Slamin, 2000, On the monotonicity of minimum diameter with respect to order and maximum out-degree, *Proceedings of COCOON 2000, Lecture Notes in Computer Science* 1858 (D.-Z. Du, P. Eades, V. Estivill-Castro, X. Lin (eds.)) (2000) 193-201.

[14] Plesnik, J. and Znam, S. 1974. Strongly geodetic directed graphs, *Acta F. R. N. Univ. Comen. - Mathematica* XXIX, (1974) 29-34.

AN APPLICATION OF ARIMA TECHNIQUE IN DETERMINING THE RAINFALL PREDICTION MODELS OVER SEVERAL REGIONS IN INDONESIA

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Abstract. In this present study, we mainly concerned an application of ARIMA (*Auto-Regressive Integrated Moving Average*) technique in determining the rainfall prediction models over several regions in Indonesia. They are Lampung (South Sumatera), Pandeglang (West Java), Indramayu (West Java), Banjarbaru (South Kalimantan), and Sumbawa Besar (Nusa Tenggara Barat). We used the monthly of surface rainfall data for period of January 1970 to December 2000. This study is motivated by the importance of understanding the mechanism of air and sea interaction between Monsoon and *El-Niño* event when they come simultaneously as already recommended by the IPCC (*Intergovernmental Panel on Climate Change* (*IPCC*) AR (*Assessment Report*) 4 and GEOSS (*Global Earth Observation System to System*). By applying the *Power Spectral Density* (PSD) and *Wavelet* analysis on those rainfall anomalies data, and also the Monsoon global index data, represented by the AUSMI (*Australian Monsoon Index*) and WNPMI (*Western North Pacific Monsoon Index*), we found a pre-dominant peak oscillation of them is about 12 month period that we call as the *Annual Oscillation* (AO). While, for the *El-Niño* event, represented by *SST* (*Sea Surface Temperature*) *Niño 3.4,* we found 60 month period. Furthermore analysis, we found more significant relationship between rainfall anomalies and AUSMI, WNPMI, and *SST Niño 3.4*. Please note here, this study was undertaken with assumption that Monsoon and *El-Niño* is interacted each other, and the model is developed by using multivariate regression method that formulated by Rainfall = a + b [AUSMI] + c [WNPMI] + d [SST Niño3.4]. By applying the ARIMA technique, we found the rainfall prediction models. They ARIMA $(1,0,1)^{12}$, with model equation $Z_t = 0.9989Z_{t-12} - 0.9338a_{t-12} + a_t$ (for AUSMI), ARIMA $(1,1,1)^{12}$, with model equation $Z_t = -0.0674Z_{t-12} + Z_{t-12} - Z_{t-24} - 0.9347a_{t-12} + a_t$, (for WNPMI), and ARIMA (2,0,2), with model equation $Z_t = 3.594Z_{t-1} - 0.8362Z_{t-2} - 1.634a_{t-1} - 0.1053a_{t-2} + a_t$ (for SST Niño 3.4). By applying these techniques, we can predict the rainfall behavior over those area until the end of 2013.

Keywords: ARIMA Technique and Rainfall Prediction Models

References

[1] Harijono, S.W.B. 2008. Analisis Dinamika Atmosfer di Bagian Utara Ekuator Sumatera Pada Saat Peristiwa *El-Niño* dan *Dipole Mode Positif* Terjadi Bersamaan, *Jurnal Sains Dirgantara* (JSD), 5(2), 130 – 148.

[2] [2]IPCC. 2001 .Climate Change 2001 : Impact, Adaptation and Vulnerabi1ity. Cambridge: Cambridge University Press.

[3] [3] Box, G. E. P., Jenkins, G. M., and Reinsel, G. C. (1994). Time Series Analysis, Forecasting and Control, 3rd ed. Prentice Hall, Englewood Clifs, NJ.

[4] [4]Chatfield, C. (1996). The Analysis of Time Series, 5th ed., Chapman & Hall, New York, NY.

[5] [5]Brockwell, Peter J. and Davis, Richard A. (1987). Time Series: Theory and Methods, Springer-Verlang..

[6] [6]Brockwell, Peter J. and Davis, Richard A. (2002). Introduction to Time Series and Forecasting, 2nd. ed., Springer-Verlang.

[7] [7]Ljung, G. and Box, G. (1978). "On a Measure of Lack of Fit in Time Series Models", Biometrika, 65, 297-303.

[8] [8]Hasan, M. Iqbal. 2003. Pokok-Pokok Materi Statistik 2 (Statistik Iterensif) Edisi Kedua. Jakarta: Bumi Aksara.

MAC WILLIAMS THEOREM FOR POSET WEIGHTS

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Abstract. In this paper we introduce the notion of local-global property. We show that a certain matrix group has a local global property and this leads for the extension theorem of the poset weights. We prove that the poset wieght over Frobenius ring satisfies the MacWilliams extension theorem if and only if the poset is a hierarchical poset.

Key words and Phrases: MacWilliams extension theorem, partitions, poset structures.

References

[1] Brualdi, R. A., Graves, J. S.\& Lawrence, K. M. (1995). Codes with a poset metric. *Discrete Mathematics*, 147(1), 57-72.

[2] Claasen, H. L., & Goldbach, R. W. (1992). A field-like property of finite rings. *Indagationes Mathematicae*, 3(1), 11-26.

[3] Goldberg, D. Y. (1980). A generalized weight for linear codes and a Witt-MacWilliams theorem. *Journal of Combinatorial Theory*, Series A, 29(3), 363-367.

[4] Hirano, Y. (1997). On admissible rings. Indagationes Mathematicae, 8(1), 55-59.

[5] Honold, T. (2001). Characterization of finite Frobenius rings. Archiv der Mathematik, 76(6), 406-415.

[6] MacWilliams, F. J. (1962). *Combinatorial problems of elementary abelian groups* (Doctoral dissertation, Radcliffe College).

[7] Niederreiter, H. (1991). A combinatorial problem for vector spaces over finite fields. *Discrete Mathematics*, 96(3), 221-228.

[8] Ward, H. N., & Wood, J. A. (1996). Characters and the equivalence of codes. *Journal of Combinatorial Theory*, Series A, 73(2), 348-352.

[9] Wood, J. A. (1999). Duality for modules over finite rings and applications to coding theory. *American journal of Mathematics*, 121(3), 555-575.

[10] Wood, J. (2008). Code equivalence characterizes finite Frobenius rings. *Proceedings of the American Mathematical Society*, 136(2), 699-706.

PARALEL SESSIONS

ON FINITE MONOTHETIC DISCRETE TOPOLOGICAL GROUPS OF PONTRYAGIN DUALITY

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Abstract. A group which has a dense cyclic subgroup plays fundamental role in theory of topological groups. Van Dantzig introduced such group as monothetic groups. Pontryagin duality is continuous homomorphism that relate some topological groups with circle groups T, that is $\mathrm{Hom}(G,T)$. Set that contains all the continuous homomorphism called by Pontryagin Dual Groups or just Dual Groups, and denoted by G_b i.e. $G_b = \{\varphi \mid \varphi : G \to T, \varphi$ continuous homomorphism $\}$.
This paper study the dual groups structure when G is monothetic and finite. It will be shown that monothetic and its dual have same order and they're isomorphic.

Key words and Phrases: circle group, cyclic, dual group, monothetic, pontryagin duality.

Abstrak. Dalam teori grup topologi, grup yang memiliki subgrup siklik yang padat menjadi hal yang sangat mendasar. Van Dantzig memperkenalkan grup dengan struktur tersebut sebagai grup monotetik. Dualitas Pontryagin merupakan homomorfisma kontinu antara grup topologi G dan grup lingkaran T, yakni $\mathrm{Hom}(G,T)$, dan himpunan semua homomorfisma tersebut membentuk grup dual, disimbolkan G_b, dengan $G_b = \{\varphi \mid \varphi : G \to T, \varphi$ homomorfisma kontinu $\}$.
Penelitian ini mengkaji struktur grup dual bilamana G adalah grup monotetik dan berhingga. Akan diperlihatkan G dan grup dualnya memiliki orde yang sama dan keduanya isomorfik.

Kata kunci: dualitas pontryagin, grup dual pontryagin, grup lingkaran, monothetic, siklik.

1. Introduction

Cyclicity concept in Group Theory has developed by using number theory as the foundation of its structure. Gauss' *Disquisitiones Arithmaticae* in 1801, shows that the set of all nonzero integers modulo prime p, each of its element is representation of every element in the set. Later, the set is known by cyclic group Zp. Gallian [1] define the cyclic group in general, a group G is said to be cyclic if there exist an element $a \in G$ so that $G = \{a^n \mid n \in Z\}$. The element a is called generator and n is order of G if n is the least integer which satisfied $a^n = \mathbf{e}$, that is an identity element of G.

Topological groups is topological space which also has group structure. A topological group G is said to be monothetic if there exist a dense cyclic subgroup H in G. The generator element of cyclic subgroup H is also the generator of monothetic group G. Falcone et al. [2] said that a group that has a dense cyclic subgroup plays a fundamental role in theory of discrete topological groups.

The circle group of complex number with modulus 1 is a multiplicative group with identity 1, denoted by T, that is $T = \{z \mid |z| = 1, z \in C\}$. Armacost [3] stated that circle group T is a monothetic group, and every its proper subgroup is isomorphic to finite cyclic group Zn of addition group modulo integers n. The circle group T is also isomorphic to quotient group R/Z with addition operation.

The Pontryagin Duality was named after Lev Semenovich Pontryagin who laid down the foundation of topological groups, specially locally compact abelian groups and their duality. The Pontryagin duality relates some topological groups G and circle groups T by continuous homomorphism, that is $Hom(G,T)$. The set of all continuous homomorphism which satisfy the duality is called Pontryagin Dual Groups or Dual Groups, denoted by G_b.

This paper discuss about Pontryagin Duality and the dual groups of monothetic groups and circle groups. The main structure of monothetic groups that has a dense cyclic subgroup will affect on the duality to circle groups. Cyclic subgroup which generated by single element will show that the images of all elements from monothetic groups will completely determine the isomorphism between the monothetic group and its dual.

2. Discrete Topological Groups and Continuous Homomorphism

All terms, definitions and theorems related with topology could be found in Mendelson [4] and Aliprantis & Burkinshaw [5]. In general, topological space is a set that contains subset from its underlying set with some properties.

Definition 2.1. *Let X be a nonempty set. A collection τ of subsets of X is said to be a topology on X if satisfies the following properties*
(i) $\emptyset \in \tau$ *and* $X \in \tau$
(ii) *For every* $U,V \in \tau$, *then* $U \cap V \in \tau$
(iii) *If* $\{V_i \mid i \in I\}$ *is a family members in* τ, *then* $\cup_{i\in I} V_i \in \tau$

If τ is a topology in X, then an ordered pair of (X,τ) is to be a topological space, and the element of τ is called open subset. The topology τ is said to be *discrete* if τ consists of all subsets from X, or $\tau = P(X)$, that is the power set of X. A *neighborhood* of x, denoted by N_x is a open subset that contain x. A topological space become *Hausdorff Space* if it satisfy the *Hausdorff Separated Axiom*, that is, for each pair of distinct points $x,y \in X$, there are neighborhood N_x and N_y such that $N_x \cap N_y = \emptyset$. A point x in subset A of X is called a closure point if every of its neighborhood contains at least one point from A, or $N_x \cap A$ 6= $\emptyset$. A set of all

closure points in A is to be a *closure* of A, symbolized by $\bar{A}$. A subset A of X is to be *dense* if $\bar{A} = X$. It is obvious that closure is a closed set. Mendelson [4] assert that the closedness of subset A assured by its density.

Theorem 2.2. *A is closed if and only if* $A = \bar{A}$.

Proof. We know that $\bar{A}$, is closed, so if $\bar{A} = A$, then A is closed. Conversely, suppose

A is closed. In this event, A itself a closed set containing A, so, therefore $\bar{A}, \subset A$. On the other hand, for an arbitrary subset A we have $A \subset \bar{A}$, for if $x \in A$, then each neighborhood N_x of x contains a point of A, namely x itself. Thus, if A is closed, $A = \bar{A}$.

Topological space that has a group structure is said to be a topological groups. These structure were connected by algebra properties of the group structure that affect on its topology structure, and vice versa. Next is definition of topological groups written by Hewitt and Ross [6].

Definition 2.3.

Let G be a set that is a group and also a topological space. Suppose that:
(i) the mapping (x,y) 7→ xy *of* $G \times G \to G$ *is a continuous*
(ii) the mapping x 7→ x^{-1} *of* $G \to G$ *is a continuous. Then G is called a topological group.*

A discrete topological space which also has a group structure is called discrete topological group. The discreteness of a topology closely related with Hausdorff property. A discrete topological group that has two distinct points will assure the availability of their disjoint neighborhood, since the discreteness will render all possibilities of open subset in its topology.

Continuous Homomorphism

This part will started with definition of continuous function between two topological spaces.

Definition 2.4. *A function* $f : (X,\tau) \to (Y,\tau_1)$ *is said to be continuous at a point* $a \in X$ *if for each neighborhoood* $N_{f(a)}$ *of* $f(a)$, $f^{-1}(N_{f(a)})$ *is neighborhood of a. The function f is said to be* continuous *if f is continuous at each point of X.*

An open subset that contained in topological space will mapped to open subset too by continuous function.

Theorem 2.5. *A function* $f : (X,\tau) \to (Y,\tau_1)$ *is continuous if and only if for each open subset O of Y ,* $f^{-1}(O)$ *is an open subset of X.*

Proof. First, suppose that f is continuous and that O is an open subset of Y . For each $a \in f^{-1}(O)$, O is a neighborhood of $f(a)$ therefore $f^{-1}(O)$ is neighborhood of a.

Since $f^{-1}(O)$ is neighborhood for each of its point, $f^{-1}(O)$ is an open subset of X. Conversely, suppose that for each open subset O in Y , $f^{-1}(O)$ is an open subset of X. Let $a \in X$ and neighborhood $N_{f(a)}$ of $f(a)$. $N_{f(a)}$ contains an open subset
O containing $f(a)$, so $f^{-1}(N_{f(a)})$ contains $f^{-1}(O)$ containing a. Thus, $f^{-1}(N_{f(a)})$ is a neighborhood of a and f is continuous at a. Since a was arbitrary in X, f is continuous.

We could use theorem 2.5 to prove the continuous function in case of closed set, since the complement of open subset O that is O^c is closed.

Homomorphism function shows the property of image, therefore we can conclude property of the domain. Note that, the operation in both groups domain and the image could be different. Next, the definition of homomorphism from Gallian[1].

Definition 2.6. *A homomorphism φ from group G to G^0 is a mapping that preserves the operation, that is $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a,b \in G$.*

If there exist two groups h$G,*$i and h$G^0,\circ$i, then homomorphism φ denoted by $\varphi(a * b) = \varphi(a) \circ \varphi(b)$. If φ is a bijection mapping, then φ is said to be an isomorphism, and both groups are called isomorphic, symbolized by G ~= G^0. The bijection mapping is a one to one and onto mapping, that is φ is said to be one to one if whenever a 6= b then $\varphi(a)$ 6= $\varphi(b)$, and said to be onto if for each $\varphi(a) \in G^0$ there is a $\varphi^{-1}(a) \in G$. The continuous homomorphism is a continuous function that preserves the operation in the group.

3. Image of Monothetic and The Dual Groups

Dikranjan [7] explained that discrete Hausdorff topological group is closed. In this case, monothetic group will satisfy such properties. By continuous function, the image of monothetic group will also closed.

Lemma 3.1. *Let φ be a continous function and M be a monothetic group. If M closed then $\varphi(M)$ is closed.*

Proof. Let consider a point z /$\in M$. Since M is discrete topological group, then there exist a neighborhood N_z thus $N_z \cap M = \emptyset$. From definition 2.4, that if φ continuous then there exist $\varphi(N_z)$ that is a neighborhood of $\varphi(z)$. For all arbitrary point z /$\in M$, we can conclude that $\cup_{\varphi(z)\in/\varphi(M)} N_{\varphi(z)} = \varphi(M)^c$ is open. So, $\varphi(M)$ is closed.

Every element in M will mapped onto T by continuous function. Every subgroup in M will also become a subgroup in T by continuous mapping. As the consequence from lemma 3.1 that explained the image of monothetic group was closed, so that the image of subgroup of monothetic will be closed.

Corollary 3.2. *Let M be a monothetic group and H its subgroup which $H \subseteq M$, if H is closed then $\varphi(H)$ is closed, by φ continuous.*

Proof. Since M is closed, so does H. Consider a point z /$\in H$, or $z \in H^c$, then there exist neighborhood $N_z \subseteq M$ but $N_z \cap H = \emptyset$. So that for $\varphi(z) \in$ T, there exist $N_{\varphi(z)} \subset$ T. Because of $N_{\varphi(z)}$ is open, and z is arbitrary point in H^c, so $\varphi(H)^c$ is union of all neighborhood, that is also an open subset in T. Since $\varphi(H)^c$ open, so $\varphi(H)$ is closed.

Suppose that M to be a monothetic group with generator m. By continuous homomorphism φ, image of m, namely $\varphi(m)$ become generator of $\varphi(M)$.

Lemma 3.3. *If M is a monothetic group with generator m, then $\varphi(M)$ will generated by $\varphi(m)$.*

Proof. Since φ is continuous, every distinct element in group will be mapped to distinct element too. Consider that the generator should be singleton, so the image will be singleton too. Note that H is cyclic subgroup in M, with generator m, that is $\langle m\rangle = H$, then $H = \langle m\rangle$. Because of H is dense in M, or $\overline{H} = M$, then lemma 3.1 assert that $\varphi(H)$ is closed with $\varphi(\overline{H}) = \langle\varphi(m)\rangle$. Since $\varphi(H)$ dense in $\varphi(M)$, then $\varphi(m)$ generate $\varphi(M)$.

Both lemmas and corollary implicitly explained the properties of monothetic's image. The following theorem will assert them.

Theorem 3.4. *Let M be a monothetic discrete topological groups, $\varphi(M)$ is monothetic if and only if φ is continuous function.*

Proof. If φ continuous, then $\varphi(M)$ monothetic. Clearly that lemma 3.1 shows the closedness of $\varphi(M)$, and corollary 3.2 state that subgroup H of M mapped to subgroup $\varphi(H)$ of $\varphi(M)$. Lemma 3.3 explain that $\varphi(m)$ is generator if m is generator in M. By continuous function φ, M is monothetic. Conversely, theorem 2.5's proof will stated that closed set will mapped to closed set by continuous function in two topological spaces. Since $\varphi(M)$ is closed then $\varphi(M)^c \subset \mathbb{T}$ is open. This is similar when M is closed, then M^c is open, with $\varphi : M^c \to \varphi(M)^c$. Because of $\varphi(M)^c$ is open and contain neighborhood for every point inside, then we can conclude that $\varphi(M)^c = \cup N_{\varphi(z)}, \forall z \notin M$. For an arbitrary point $z \notin M$, then $\varphi^{-1}(\cup N_{\varphi(z)}) = M^c$, that surely open. According to definition 2.4 and theorem 2.5, then φ is continuous.

Pontryagin Dual Groups Structure

Suppose that M be a monothetic group with cyclic subgroup H. Then H become closed subgroup which generated by an element m, and the order of H suppose w, so that m^w is the identity element in H. Consider that an element m in H will mapped to $z \in \mathbb{T}$.

$$\varphi(m) = z \in \mathbb{T}$$

Since m^w is identity in H, by trivial continuous mapping, $\varphi(m^w) = 1$, with 1 is the identity of circle group $\mathbb{T}$. Lemma 4.2 state that image of generator will be generator too,

$$\begin{aligned} m &\mapsto \varphi(m) \\ \varphi(m)^w &= 1 \in \varphi(H) \\ zw &= 1 \end{aligned}$$

Note that the homomorphism φ determined by $z \in \varphi(H)$ which satisfy $z^w = 1$. Previously, we know that $\mathbb{T}$ is isomorphic with quotient group of addition $\mathbb{R}/\mathbb{Z}$, that is a representative of all real numbers in $[0,1)$. Since each element $z = e^{i\alpha} \in \mathbb{T}$ be representated by $\alpha \in [0,2\pi)$, then α should be convert to real numbers $[0,1)$ by continuous function f i.e. $f : [0,1) \to [0,2\pi)$.

$$f : [0,1) \to [0,2\pi)$$
$$f(a) = a2\pi$$

with $a \in [0,1)$ and $\alpha \in [0,2\pi)$. For each $z = e^{ia2\pi}$ and w an order of H, we could form

$\frac{k}{w} \in [0,1)$ with $0 \le k \le w$. So that $z = e^{ik\frac{2\pi}{w}}$, and consequently,

$z^w = e^{(ik\frac{2\pi}{w})w} = e^{i2\pi k}$, by Euler Identity that state $e^{i\pi} = -1$, then $e^{i\pi(2k)} = 1$. In other word, for all $z \in \varphi(H) \subset$ T representing all homomorphism between H and $\varphi(H) \subset$ T. So, the dual group H can be written as

$H_b = \{\varphi \mid \varphi : m \mapsto z, \langle m \rangle = H, \forall z \in \varphi(H) \subset T, \varphi$ continuous homomorphism$\}$

Beside that, the number of element in H is equal to H_b, the notation would be $||H|| = ||H_b||$. Since H dense in M, it certainly satisfied that $||M|| = ||M_c||$.

Theorem 3.5. *If M is monothetic group, then M_c is monothetic group.*

Proof. Firstly, it will show that M_c satisfy the definition of topological group. Suppose that continuous homomorphism $\varphi_a, \varphi_b \in M_c$ and m be the generator of M. Note that φ_a dan φ_b is continuous homomorphism in M_c that bring $m \in M$ to a and b di T. The Ordered pair (φ_a, φ_b) that contained in $M_c \times M_c$ can be mapped by ψ to M_c, that is $\psi : (\varphi_a, \varphi_b) \mapsto (\varphi_a\varphi_b)(m)$.

$\psi : M_c \times M_c \to M_c$

$$\psi : (\varphi_a, \varphi_b) \mapsto (\varphi_a\varphi_b)(m) \quad (\varphi_a\varphi_b)(m) = \varphi_a(m)\varphi_b(m)$$

Since $a \in$ T has an inverse $a^{-1} \in$ T, then ψ bring φ_a to its inverse, that is a function which mapped to a^{-1}. It can written as follows, $\psi : \varphi_a \mapsto \varphi_{a^{-1}}$. So that M_c is topological group. Next, M_c is monothetic group if it contain a dense cyclic subgroup H_b. This can be proved by showing the generator of M_c. If m generator of M and $\varphi(m)$ generator of $\varphi(M)$, then m and $\varphi(m)$ also a generator of H and $\varphi(H)$, respectively. So there exist $\varphi_1 \in M_c$ which bring m to $\varphi(m)$. The operation in φ_m will depend on $\varphi_m(m)$. If $\varphi_m(m)$ is generator of $\varphi(M)$, it can written

$$\begin{aligned}
\phi_m\phi_m &= \phi_m(m)\phi_m(m) &&= (\phi_m(m))^2 \\
\phi_m\phi_m\phi_m &= \phi_m(m)\phi_m(m)\phi_m(m) &&= (\phi_m(m))^3 \\
&\vdots \\
\underbrace{\phi_m\phi_m\cdots\phi_m}_{w-tuple} &= \phi_m(m)\phi_m(m)\cdots\phi_m(m) &&= (\phi_m(m))^w
\end{aligned}$$

If w the order of M and $\varphi(M)$, then $\varphi_1(m)^w = 1 \in \varphi(M)$ that is the identity of $\varphi(M)$. Since theorem 2.4 assert that the homomorphism which mapped to identity is homomorphism identity, then $\phi_m{}^w = \phi_e$. So, φ_m that bring $m \in M$ to $\varphi(m) \in \varphi(M)$ is the generator of M_c.

Note that in this case, monothetic group has same number of element with its dual group. Next, the bijection between M dan M_c will be shown, so that both of them are isomorphic.

Theorem 3.6. *If M is a monothetic group, then $M \cong M_c$*

Proof. To show the isomorphism between M and its dual, let ϕ be a continuous function. Suppose that the order of M and M_c is w. The binary operation on M is $*$ that is for $m_a, m_b \in M$, $0 \le a,b \le w$ with $m_a * m_b \in M$. The continuous function ϕ will define as $\phi(m_a * m_b) = (\varphi_a\varphi_b)(m)$. If we set $m_b = \mathbf{e}$ as the identity, then

$$\begin{aligned}
\phi &: M \to M_c \\
\phi(m_a * \mathbf{e}) &= (\varphi_a\varphi_e)(m) \\
\phi(m_a) &= (\varphi_a)(m)
\end{aligned}$$

The similar case will occur if we set $m_a = \mathbf{e}$. So, for $m_a \neq m_b$ obtained $(\varphi_a)(m) \neq (\varphi_a)(m)$, that is ϕ is one to one function. Next, for every $\varphi_a(m)$, there exist

$\phi^{-1}(\varphi_a(m)) = m_a$, which shows ϕ is onto. Because ϕ is one to one and onto, then the bijection has been satisfied. For every $m_a * m_b \in M$ occur

$$\begin{aligned}\phi(m_a * m_b) &= (\varphi_a\varphi_b)(m)\\ &= \varphi_a(m)\varphi_b(m)\\ &= \phi(m_a)\phi(m_b)\end{aligned}$$

So that ϕ is homomorphism, which means ϕ is isomorphism, or $M \cong M_c$.

4. Concluding Remarks

Let M be a monothetic group, and M_c is Pontryagin Dual Groups, then the conclusion as follows.

1. By continuous homomorphism φ, image of monothetic group in circle group also monothetic. If m generate M, the $\varphi(m)$ generate $\varphi(M) \subseteq$ T. If $H \subseteq M$ is cyclic subgroup, then $\varphi(H) \subseteq \varphi(M)$ is also a cyclic subgroup.
2. The number of homomorphism $\varphi \in M_c$ determined by mapping of generator m. Cyclic subgroup H is isomorphic with H_b, it also occur to $M \cong M_c$.

References

[1] Gallian, J. A., *Contemporary Abstract Algebra 7th*, Belmont: Brook/Cole, (2010).

[2] Falcone, G., Plaumann, P., Strambach, K., Monothetic Algebraic Groups, *J. Aust. Math. Soc.* , Vol. **82**, pp. 315-324, (2007).

[3] Armacost, D. L., *The Structure of Locally Compact Abelian Groups*, New York: Marcel Dekker, Inc., (1981).

[4] Mendelson, B., *Introduction to Topology 3rd*, Boston: Allyn and Bacon, Inc., (1975).

[5] Aliprantis, C. D. & Burkinshaw, O., *Principles of Real Analysis 2nd*, London: Academic Press, Inc , (1990).

[6] Hewitt, E. & Ross, K. A., *Abstract Harmonic Analysis 2nd*, New York: Springer-Verlag,

[7] (1979).

[8] Dikranjan, D. Introduction to Topological Groups. *Topologia 2*. (2013).

LINEAR INDEPENDENCE OVER THE SYMMETRIZED MAX PLUS ALGEBRA

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Abstract. The symmetrized max plus algebra is an algebraic structure which is a commutative semiring, has a zero element $\epsilon = -\infty$, the identity element e = 0, and an additively idempotent. Motivated by the the previous study as in conventional linear algebra , in this paper will be described the necessary condition of linear independence over the symmetrized max plus algebra.We show that if a the columns of a matrix over the symmetrized max plus algebra are linearly dependent, then the determinat of that matrix is

Key words and Phrases: the symmetrized max plus algebra, linear independence, determinant

Abstrak. Aljabar maks plus tersimetris adalah suatu struktur aljabar yang merupakan semiring komutatif, mempunyai elemen nol $\epsilon = -\infty$, elemen identitas e = 0 dan idempoten penjumlahan. Termotivasi seperti dalam aljabar linier konvensional, dalam makalah ini akan dikembangkan syarat perlu bebas linier atas aljabar maks plus tersimetris. Akan ditunjukkan bahwa jika kolom dari matriks atas aljabar maks plus tersimetris adalah tak bebas linier maka determinan matriks tersebut adalah

.Kata kunci: aljabar maks plus tersimetris, bebas linier, determinan

1. Introduction

The system max plus algebra lacks an additive inverse. Therefore, some equations do not have a solution. For example, the equation $3 \oplus x = 2$ has no solution since there is no x such that $\max(3,x) = 2$. In Schutter (1997) and Singh (2008) state that one way of trying to solve this problem is to extend the max plus algebra to a larger system which will include and additive inverse in the same way we have that the system $(\mathbb{S}, \oplus, \otimes)$ is called the symmetrized max plus algebra and $\mathbb{S} = \mathbb{R}_\epsilon^2/\mathcal{B}$ with $\mathcal{B}$ is an equivalence relation. Motivated by linear dependence dan independence over the conventional algebra, in this paper will be described linear independence over the symmetrized max plus algebra. Akian et al. (2009) state that a family $m_1, m_2, ..., m_k$ of elements of a semimodule M over a semiring S is linearly dependent in the Gondran-Minoux sense if there exist two subset $I, J \subseteq K := 1, 2, ..., k, I \cap J = \emptyset, I \cup J = K$, and scalars $\alpha_1, \alpha_2, ..., \alpha_k \in S$, not all equal to 0, such that $\sum_{i \in I} \alpha_i m_i = \sum_{j \in J} \alpha_j m_j$.

2. **The Symmetrized Max Plus Algebra**

For the set of all real numbers $\mathbb{R}_\epsilon = \mathbb{R} \cup \{\epsilon\}$ with

$$\epsilon := -\infty \ and \ e := 0.$$

For all $a, b \in \mathbb{R}_\epsilon$, the operations $\oplus$ and $\otimes$ are defined as follows:

$$a \ \oplus \ b \ = \ max \ (a, b) \ and \ a \ \otimes \ b \ = \ a \ + \ b$$

and then,$(\mathbb{R}_\epsilon, \oplus, \otimes$) is called the max plus algebra.

Definition 2.1. *(Schutter (1996)) Let $u = (x, y), v = (w, z) \in \mathbb{R}_\epsilon^2$.*
1. Two unary operators $\ominus$ and $(.)^\bullet$ are defined as follow :

$$\ominus u = (y, x) \ and \ u^\bullet = u \oplus (\ominus u)$$

2. An element u is called balances with v, denoted by $u \ \nabla \ v$, if

$$x \oplus z = y \oplus w.$$

3. A relation $\mathcal{B}$ is defined as follows :

$$(x, y)\mathcal{B}(w, z) \ if \ \begin{cases} (x, y)\nabla(w, z), \ if \ x \neq y \ and \ w \neq z \\ (x, y) = (w, z), \ otherwise \end{cases}$$

Because $\mathcal{B}$ is an equivalence relation, we have the set of factor $\mathbb{S} = \mathbb{R}_\epsilon^2/\mathcal{B}$ and the system $(\mathbb{S}, \oplus, \otimes)$ is called the symmetrized max plus algebra, with the operations of addition and multiplication on $\mathbb{S}$ is defined as follows
$\overline{(a, b)} \oplus \overline{(c, d)} = \overline{(a \oplus c, b \oplus d)}$
$\overline{(a, b)} \otimes \overline{(c, d)} = \overline{(a \otimes c \oplus b \otimes d, a \otimes d \oplus b \otimes c)}$, for $\overline{(a, b)}, \overline{(c, d)} \in \mathbb{S}$
The system $(\mathbb{S}, \oplus, \otimes)$ is a semiring, because:

(1) $(\mathbb{S}, \oplus)$ is associative
(2) $(\mathbb{S}, \otimes)$ is associative
(3) $(\mathbb{S}, \oplus, \otimes)$ satisfies both the left and right distributive

Lemma 2.2. *(Schutter (1996)) Let $(\mathbb{S}, \oplus, \otimes)$ be the symmetrized max plus algebra. Then the following statements holds.*

(1) $(\mathbb{S}, \oplus, \otimes)$ is commutative
(2) An element $\overline{(\epsilon, \epsilon)}$ is a zero element and an absorbent element.

(3) *An element* $\overline{(e,\epsilon)}$ *is an identity element.*
(4) $(\mathbb{S},\oplus,\otimes)$ *is an additively idempotent.*

The system $\mathbb{S}$ is divided into three classes :

- $\mathbb{S}^{\oplus}$ consists of all positive elements or
  $\mathbb{S}^{\oplus}= \{\overline{(t,\epsilon)}|t \in \mathbb{R}_\epsilon\}$ with $\overline{(t,\epsilon)} = \{(t,x) \in \mathbb{R}^2_\epsilon | x < t\}$
- $\mathbb{S}^{\ominus}$ consists of all negative elements or
  $\mathbb{S}^{\ominus}= \{\overline{(\epsilon,t)}|t \in \mathbb{R}_\epsilon\}$ with $\overline{(\epsilon,t)} = \{(x,t) \in \mathbb{R}^2_\epsilon | x < t\}$
- $\mathbb{S}^{\bullet}$ consists of all balanced elements or
  $\mathbb{S}^{\bullet}= \{\overline{(t,t)}|t \in \mathbb{R}_\epsilon\}$ with $\overline{(t,t)} = \{(t,t) \in \mathbb{R}^2_\epsilon\}$

Because $\mathbb{S}^{\oplus}$ isomorfic with $\mathbb{R}_\epsilon$, so it will be shown that for $a \in \mathbb{R}_\epsilon$, can be expressed by $\overline{(a,\epsilon)} \in \mathbb{S}^{\oplus}$.
Furthermore, it is easy to verify that for $a \in \mathbb{R}_\epsilon$ we have :

$$a = \overline{(a,\epsilon)} \ with \ \overline{(a,\epsilon)} \in \mathbb{S}^{\oplus}$$

$$\ominus a = \ominus\overline{(a,\epsilon)} = \overline{\ominus(a,\epsilon)} = \overline{(\epsilon,a)} \ with \ \overline{(\epsilon,a)} \in \mathbb{S}^{\ominus}$$

$$a^{\bullet} = a \ominus a = \overline{(a,\epsilon)} \ominus \overline{(a,\epsilon)} = \overline{(a,\epsilon)} \oplus \overline{(\epsilon,a)} = \overline{(a,a)} \in \mathbb{S}^{\bullet}$$

Lemma 2.3. *For* $a,b \in \mathbb{R}_\epsilon, a \ominus b = \overline{(a,b)}$.

PROOF.

$$a \ominus b = \overline{(a,\epsilon)} \ominus \overline{(b,\epsilon)} = \overline{(a,\epsilon)} \oplus \overline{(\epsilon,b)} = \overline{(a,b)}$$

Lemma 2.4. *For* $\overline{(a,b)} \in \mathbb{S}$ *with* $a,b \in \mathbb{R}_\epsilon$*, the following statements hold :*

(1) *If* $a>b$ *then* $\overline{(a,b)} = \overline{(a,\epsilon)}$
(2) *If* $a<b$ *then* $\overline{(a,b)} = \overline{(\epsilon,b)}$
(3) *If* $a=b$ *then* $\overline{(a,b)} = \overline{(a,a)}$ *or* $\overline{(a,b)} = \overline{(b,b)}$

PROOF.

(1) For $a > b$ we have that $a \oplus b = a$. In other words, $a \oplus \epsilon = a \oplus b$. The result that $(a,b)\nabla(a,\epsilon)$. So it follows that $(a,b)\mathcal{B}(a,\epsilon)$. Therefore $\overline{(a,b)} = \overline{(a,\epsilon)}$.
(2) For $a < b$ we have that $a \oplus b = b$. In other words, $a \oplus b = b \oplus \epsilon$. The result that $(a,b)\nabla(\epsilon,b)$. So it follows that $(a,b)\mathcal{B}(\epsilon,b)$. Therefore $\overline{(a,b)} = \overline{(\epsilon,b)}$.

Corollary 2.5. *For* $a,b \in \mathbb{R}_\epsilon$,

$$a \ominus b \ = \begin{cases} a, \ if \ a > b \\ \ominus b, \ if \ a < b \\ a^{\bullet}, \ if \ a = b \end{cases}$$

3. **Matrices over The Symmetrized Max Plus Algebra**

Let $\mathbb{S}$ the symmetrized max plus algebra, n a positive integer greater than 1 and $M_n(\mathbb{S})$ is the set of all nxn matrices over $\mathbb{S}$. Operation $\oplus$ and $\otimes$ for matrices over the symmetrized max plus algebra is defined :
$C = A \oplus B \;\Rightarrow\; c_{ij} = a_{ij} \oplus b_{ij}$
$C = A \otimes B \;\Rightarrow\; c_{ij} = \bigoplus_l a_{il} \otimes b_{lj}$
Zero matrix nxn over $\mathbb{S}$ is ϵ_n with $(\epsilon_n)_{ij} = \epsilon$ and identity matrix nxn over $\mathbb{S}$ is E_n with

$$[E_n]_{ij} \;=\; \begin{cases} e, \; jika \; i = j \\ \epsilon, \; jika \; i \neq j \end{cases}$$

Definition 3.1. *We say that the matrix* $A \in M_n(\mathbb{S})$ *is invertible over* $\mathbb{S}$ *if*

$$A \otimes B \nabla E_n \; dan \; B \otimes A \nabla E_n$$

for any $B \in M_n(\mathbb{S})$.

Definition 3.2. *Let a matrix* $A \in M_n(\mathbb{S})$. *The determinant of* A *is defined by*

$$det \; A = \bigoplus_{\sigma \in \mathcal{S}_n} sgn(\sigma) \otimes (\bigotimes_{i=1}^{n} A_{i\sigma(i)})$$

with S_n *is the set of all permutations of* $\{1, 2, ..., n\}$, *and*

$$sgn(\sigma) = \begin{cases} 0, \; if \; \sigma \; is \; even \; permutation \\ \ominus 0, \; if \; \sigma \; is \; odd \; permutation \end{cases}$$

Note that the operator "(∇)" and the systems of max-linear balances hold :

Lemma 3.3. *(1)* $\forall a, b, c \in \mathbb{S}, a \ominus c \nabla b \Leftrightarrow a \nabla b \oplus c$
(2) $\forall a, b \in \mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus}, a \nabla b \Rightarrow a = b$
(3) Let $A \in M_n(\mathbb{S})$. *The homogeneous linear balance* $A \otimes x \nabla \epsilon_{nx1}$ *has a non trivial solution in* $\mathbb{S}^{\oplus}$ *or* $\mathbb{S}^{\ominus}$ *if and only if* $det(A) \nabla \epsilon$.

4. **Linear Independence over The Symmetrized Max Plus Algebra**

The symmetrized max plus algebra is an idempotent semiring, so in order to define rank, linear combination, linear dependence, and independence we need definition of a semimodule. A semimodule is essentially a linear space over a semiring. Let $\mathbb{S}$ be a semiring with ϵ as a zero.
A (left) semimodule $M_{n\times 1}(\mathbb{S})$ over $\mathbb{S}$ is a commutative monoid $(\mathbb{S}, \oplus)$ with zero element $\epsilon \in M_{n\times 1}(\mathbb{S})$, together with an $\mathbb{S}-$multiplication

$$\mathbb{S} \times M_{n\times 1}(\mathbb{S}) \longrightarrow M_{n\times 1}(\mathbb{S}), (r, x) \longrightarrow r.x = r \otimes x$$

such that, for all $r, s \in \mathbb{S}$ and $x, y \in M_{n\times 1}(\mathbb{S})$, we have

(1) $r \otimes (s \otimes x) = (r \otimes s) \otimes x$

(2) $(r \oplus s) \otimes x = r \otimes x \oplus s \otimes x$
(3) $e \otimes x = x$
(4) $r \otimes \epsilon = \epsilon$
(5) $r \otimes (x \oplus y) = r \otimes x \oplus r \otimes y$

In a similar way, a right semimodule can be defined.
The rank, linear combination, and linear independence are given in the next definitions.

Definition 4.1. *(Schutter (1997)) Let $A \in M_{mxn}(\mathbb{S})$. The max-algebraic minor rank of A, $rank_{\oplus}(A)$, is the dimension of the largest square submatrix of A the max-algebraic determinant of which is not balanced.*

Definition 4.2. *Let $a_1, a_2, ..., a_n \in M_{nx1}(\mathbb{S})$ and $\alpha_1, \alpha_2, ..., \alpha_m \in \mathbb{S}$.
The expression $\bigoplus_{i=1}^{m} \alpha_i \otimes a_i$ is called a linear combination of $\{a_1, a_2, ..., a_n\}$.*

Definition 4.3. *A set of vectors $a_i \in M_{nx1}(\mathbb{S})|i = 1, 2, ..., m$ is said to be a linearly independent set whenever the only solution for the scalars α_i in $\bigoplus_{i=1}^{m} \alpha_i \otimes a_i \nabla \epsilon_{nx1}$ is the trivial solution $\alpha_i = \epsilon$.*

The relation between linear dependence and linear combination are given in the following theorem.

Lemma 4.4. *If the set of vectors $\{a_i \in M_{nx1}(\mathbb{S})|i = 1, 2, ..., n\}$ is linearly dependent then one of the vectors can be presented as a linear combination of the other vectors in the set.*

Proof.
Let $\alpha_1 \otimes a_1 \oplus \alpha_2 \otimes a_2 \oplus ... \oplus \alpha_n \otimes a_n \nabla \epsilon$.
Because $\{a_i \in M_{nx1}(\mathbb{S})|i = 1, 2, ..., n\}$ is a linearly dependent set, without loss of generality, we can take $\alpha_1 \neq \epsilon$.
So, there is a scalar $\alpha_1^{\otimes^{-1}}$ such that

$$\alpha_1^{\otimes^{-1}} \otimes (\alpha_1 \otimes a_1 \oplus \alpha_2 \otimes a_2 \oplus ... \oplus \alpha_n \otimes a_n) \nabla \alpha_1^{\otimes^{-1}} \otimes \epsilon$$
$$a_1 \oplus \alpha_1^{\otimes^{-1}} \alpha_2 \otimes a_2 \oplus ... \oplus \alpha_1^{\otimes^{-1}} \otimes \alpha_n \otimes a_n \nabla \epsilon$$
$$a_1 \nabla \beta_2 \otimes a_2 \oplus \beta_3 \otimes a_3 \oplus ... \oplus \beta_n \otimes a_n \text{ or } a_1 \nabla \bigoplus_{i=2}^{n} \beta_i \otimes a_i$$

This leads to a characterization of linear dependent in term of determinants.

Theorem 4.5. *Let $\{a_i \in M_{nx1}(\mathbb{S})|i = 1, 2, ..., n\}$ be a vector set.
Construct a matrix A such that $A = \begin{pmatrix} a_1 & a_2 & ... & a_n \end{pmatrix}$.
The set of vectors $\{a_i \in M_{nx1}(\mathbb{S})|i = 1, 2, ..., n\}$ are linearly dependent if and only if $det(A) \nabla \epsilon$.*

Proof.
($\Rightarrow$) Because $\{a_i \in M_{nx1}(\mathbb{S})|i = 1, 2, ..., n\}$ is a linearly dependent set, this implies that one of the vectors (without loss of generality, let's say its the vector a_1) can

be presented as a linear combination of the other vectors in the set. Or,

$$a_1 \nabla \beta_2 \otimes a_2 \oplus \beta_3 \otimes a_3 \oplus ... \oplus \beta_n \otimes a_n \ or \ a_1 \nabla \bigoplus_{i=2}^{n} \beta_i \otimes a_i$$

Next, take matrix A and subtract a_1 with $\bigoplus_{i=2}^{n} \beta_i \otimes a_i$. This results in another matrix (say A') whose last column is a ϵ vector.

$$A = \begin{pmatrix} a^{\bullet} & a_2 & ... & a_n \end{pmatrix} \nabla \begin{pmatrix} \epsilon & a_2 & ... & a_n \end{pmatrix}$$

Because $det \begin{pmatrix} \epsilon & a_2 & ... & a_n \end{pmatrix} \nabla \epsilon$ so, we now have that $det(A') \nabla \epsilon$.
It follows from the fact that one of the columns of the matrix being ϵ, that $det(A) \nabla \epsilon$.
($\Leftarrow$) Let $\{a_i \in M_{nx1}(\mathbb{S}) | i = 1, 2, ..., n\}$ is a linearly independent.
We can show that det(A) not balanced with ϵ.
Construct

$$\alpha_1 \otimes a_1 \oplus \alpha_2 \otimes a_2 \oplus \ldots \oplus \alpha_n \otimes a_n \nabla \epsilon$$

We have that

$$a_1 \otimes \alpha_1 \oplus a_2 \otimes \alpha_2 \oplus \ldots \oplus a_n \otimes \alpha_n \nabla \epsilon$$

Consequently, $\begin{pmatrix} a_1 & a_2 & ... & a_n \end{pmatrix} \otimes \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \nabla \epsilon$.
Because $\{a_i \in M_{nx1}(\mathbb{S}) | i = 1, 2, ..., n\}$ is a linearly independent, so we have

$$\alpha_1 = \alpha_2 = ... = \alpha_n = \epsilon$$

We can see, that homogenous linear balance $A \otimes x \nabla \epsilon$ has a trivial solution $x = \epsilon$.
Since it follows from lemma 3.3 that the homogeneous linear balance $A \otimes x \nabla \epsilon$ has a non trivial signed solution if and only if $det A \nabla \epsilon$, so we have det(A) not balanced with ϵ.

Example 1 : Let $\{v_1, v_2, v_3, v_4\} \subseteq M_{4\times1}(\mathbb{S})$ with
$v_1 = \begin{pmatrix} 0 \\ \ominus 7 \\ \epsilon \\ \ominus 2 \end{pmatrix}, v_2 = \begin{pmatrix} \ominus 2 \\ 9 \\ 5 \\ 4 \end{pmatrix}, v_3 = \begin{pmatrix} \epsilon \\ 3 \\ 6 \\ 0 \end{pmatrix}, \ and \ v_4 = \begin{pmatrix} 0 \\ \ominus 7 \\ 9 \\ 3 \end{pmatrix}$
We have, v_2 and v_4 are linear combinations from $\{v_1, v_3\}$ such that

$$v_2 = \ominus 2 \otimes v_1 \oplus (-1) \otimes v_3$$

and

$$v_4 = v_1 \oplus 3 \otimes v_3.$$

We have $det \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix} = det \begin{pmatrix} 0 & \ominus 2 & \epsilon & 0 \\ \ominus 7 & 9 & 3 & \ominus 7 \\ \epsilon & 5 & 6 & 9 \\ \ominus 2 & 4 & 0 & 3 \end{pmatrix}$
$= (18 \oplus 16 \oplus \epsilon \ominus 12) \ominus (\ominus 10 \ominus 17 \oplus 18 \oplus \epsilon) = 18 \ominus 18 = 18^{\bullet} \nabla \epsilon$.
From theorem 4.5, we have that $\{v_1, v_2, v_3, v_4\}$ is linearly dependent.

The following theorem shows that relation between non trivial solution and the rank of matrix in the homogeneous linear balance.

Theorem 4.6. *Let $A \in M_{mxn}(\mathbb{S})$.*
The homogeneous linear balance $A \otimes x \nabla \epsilon$ has a non trivial signed solution (i.e. $x \not\subseteq M_{n\times 1}(\mathbb{S}^{\bullet})$) if and only if $rank_{\oplus}(A) < n$.

PROOF.
($\Rightarrow$) Let rank(A) = n = r.
We have $A_r \otimes x \nabla \epsilon$ can be represented as
$A_r \otimes x = \begin{pmatrix} E_r \\ \epsilon \end{pmatrix} \otimes x \nabla \epsilon.$
So, the only solution of $A_r \otimes x \nabla \epsilon$ is $x \nabla \epsilon$.
($\Leftarrow$) Let $A \in M_{mxn}(\mathbb{S})$ and $rank(A) = r < n$.
Suppose A_r, the row echelon form of A can be represented as a form

$$A_r = \begin{pmatrix} e & \epsilon & \dots & \epsilon & * & \dots & * \\ \epsilon & e & \dots & \epsilon & * & \dots & * \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \epsilon & \epsilon & \dots & e & * & \dots & * \\ \epsilon & \epsilon & \dots & \epsilon & \epsilon & \dots & \epsilon \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \epsilon & \epsilon & \dots & \epsilon & \epsilon & \dots & \epsilon \end{pmatrix} = \begin{pmatrix} E_r & C \\ \epsilon & \epsilon \end{pmatrix}.$$

Without loss of generality, the rank A is r and there is at least $r+1$ columns.
Therefore, C has at least one columns and
$A_r \otimes x = \begin{pmatrix} E_r & C \\ \epsilon & \epsilon \end{pmatrix} \otimes x = \begin{pmatrix} E_r & C \\ \epsilon & \epsilon \end{pmatrix} \otimes \begin{pmatrix} z \\ y \end{pmatrix} \nabla \epsilon$. Therefore, $E_r \otimes z \oplus C \otimes y \nabla \epsilon$ or $z = \ominus C \otimes y$.
We have shown that $x = \begin{pmatrix} \ominus C \otimes y \\ y \end{pmatrix}$ is non trivial solution for y not balanced with ϵ.

Example 2 :

Let $A \otimes x \nabla \epsilon$ with $A = \begin{pmatrix} \ominus 2 & 1^{\bullet} & 0 & \epsilon \\ \epsilon & 1 & \ominus 0 & \epsilon \\ 1 & 0^{\bullet} & \epsilon & 1 \end{pmatrix}$.

With row echelon form, we have $rank(A) = 3 < 4$ and $C = \begin{pmatrix} \ominus(-2) \\ \ominus(-1) \\ (-2)^{\bullet} \end{pmatrix}$

We can show that for $y = 1$, we have $x = \begin{pmatrix} -1 \\ 0 \\ (-1)^{\bullet} \\ 1 \end{pmatrix}$ is nontrivial solution.

Theorem 4.7. *Let the set of vectors $S = \{a_i \in M_{nx1}(\mathbb{S}) | i = 1, 2, ..., n\}$.*
Construct a matrix A such that $A = \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix}$ with $m \times n$.
If $n > m$ then S is linearly dependent.

PROOF.
Let $\{a_i \in M_{nx1}(\mathbb{S}) | i = 1, 2, ..., n\}$ and construct $\alpha_1 \otimes a_1 \oplus \alpha_2 \otimes a_2 \oplus \ldots \oplus \alpha_n \otimes a_n \nabla \epsilon$
We have that $a_1 \otimes \alpha_1 \oplus a_2 \otimes \alpha_2 \oplus \ldots \oplus a_n \otimes \alpha_n \nabla \epsilon$

So, $\begin{pmatrix} a_1 & a_2 & ... & a_n \end{pmatrix} \otimes \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \nabla \epsilon.$

Because $A = \begin{pmatrix} a_1 & a_2 & ... & a_n \end{pmatrix}$ with $m \times n,\ m < n,\ and\ rank(A) \leq n.$
This implies that $rank(A) \leq m < n.$

Since it follows from Theorem 4.7, so $\begin{pmatrix} a_1 & a_2 & ... & a_n \end{pmatrix} \otimes \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \nabla \epsilon$ has a non trivial solution.
Therefore, there is $\alpha_i \neq \epsilon$.
This means that $S = \{a_i \in M_{nx1}(\mathbb{S}) | i = 1, 2, ..., n\}$ is linearly dependent.

References

[1] Akian, M., Gaubert, S., and Guterman, A., 2008. *Linear Independence over Tropical Semirings and Beyond*.
[2] Farlow, Kasie G.,2009. Max Plus Algebra, Master's Thesis submitted to the Faculty of the Virginia Polytechnic Institute and State University in partial fulfillment of the requirements for the degree of Masters.
[3] Gaubert, S., 1992.$Th\acute{e}oriedesSyst\grave{e}mesLin\acute{e}airesdanslesDio\ddot{i}des$. PhD thesis, Ecole Nationale $Sup\acute{e}rieuredesMinesdeParis, France$.
[4] Schutter, B. De., 1996. Max-Algebraic System Theory for Discrete Event Systems, PhD thesis, Faculty of Applied Sciences, K.U. Leuven, Leuven, Belgium.
[5] Schutter, B. De and Moor, B. De, 1997. A Note on The Characteristic Equation in The Max-Plus Algebra. *Linear Algebra and Its Applications*, vol. 261, no. 13, pp. 237250.
[6] Singh, D., Ibrahim, M., and Singh, J.N., 2008. A Note on Symmetrized Max-Plus Algebra. *Journal of Mathematical Sciences and Mathematics Education*. Vol. 5. No.1.

ON GRADED N- PRIME SUBMODULES

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Abstract. Let R be any ring with identity, right $R-$modules M and $S = End_R(M)$. Submodule X of M is called fully invariant submodule of M if for any $s \in S$, we have $s(X) \subset X$. Let be X is a fully invariant proper submodule of M. Then X is called a prime submodule of M if for any ideal I of S and any fully invariant submodul U of M, $I(U) \subset X$ implies $I(M) \subset X$ or $U \subset X$. In this paper we will define consept submodule from Sanh in graded $R-$module and we called graded $N-$prime submodule, Furthermore we investigate some characterization. We found out that.

Key words and Phrases: $N-$prime submodule, graded $N-$prime submodule.

1. Introduction

Dauns[4] have introduced the prime submodule and investigated some properties in 1978. Research about graded ring and graded module have introduced by C. Natasescu and F.Van Oystaeyen in 1982. In 2006, Atani[1] have introduced the graded prime submodule of graded module and investigated some properties . Also , graded prime submodule have been introduced and stadied in [2],[5].

Before we state some result, let us introduce some notations and terminologies . Let G be an abelian group with identity e and let R be any ring with identity. The ring R is called graded ring if $\mathrm{R} = \oplus_{\mathrm{g}\in\mathrm{G}}\mathrm{R}_\mathrm{g}$ where R_g is an additive subgroup of R and $\mathrm{R}_\mathrm{g}\mathrm{R}_\mathrm{h} \subseteq \mathrm{R}_{\mathrm{gh}}$ for every g, h in G. The summands R_g are called homogeneous components. If $a \in R$ then a can be written uniquely as $\sum_{g\in G} a_g$ where a_g is component of a in R_g. In this case , R_e is a subring of R and $1_R \in R_e$.

Let R be a graded ring and M an $R-$module. We say that M is a graded $R-$module if there is exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that $M = \oplus_{g\in G} M_g$ and $R_g M_h \subseteq M_{gh}$, here $R_g M_h$ denotes the additive subgroup of M consisting of all finite suma of elements $r_g s_h$ with $r_g \in R_g$ and $s_h \in S_h$. If M is a graded $R-$module, then M_g is R_e-module for all $g \in G$. Submodule N of graded $R-$module M is called graded submodule of M if $N = \oplus_{g\in G} N_g$ where $N_g = N \cap M_g$ for $g \in G$.
Sanh [7] have introduced the prime submodule of fully invariant submodule of $R-$module M. Definition of prime submodule from Sanh is base on the properties

that every R −homomorphism from R_R to R_R is a left multiplication and that $End_R(R)$ is isomorphic to R. In this paper, the prime submodule is defined by Sahn denoted by N −prime submodule.
In this paper we will define consept submodule from Sanh in graded R −module and we called graded N −prime submodule, Furthermore we will investigate some characterization.

2. Main Results

Let R be a graded ring, $M\ and\ N$ are graded R −module and $f: M \to N$ is an R −module homomorphism. Then f said to be a graded R −module homomorphism of degree k if $f(M_g) \subseteq N_{g+k}$ for each $g \in G$. Graded homomorphism without an indication of degree are understood to have degree zero. We let $(END_R(M))_k$ be the set of graded module homomorphism from M to M of degree k and let $END_R(M) = \bigoplus_{k\in G}(END_R(M))_k$. Then $END_R(M)$ is a graded ring and $END_R(R)$ is a subring of $End_R(M)$. The graded ring $END_R(R) = \bigoplus_{k\in G} END_R(R)_k$ is isomorphic to $R = \bigoplus_{k\in G} R_k$

Let M be graded right R −module and $S = END_R(M)$, its endomophism ring. Graded submodule X of M is called a fully invariant graded submodule of M if for any $s \in S$, $s(X) \subset X$. By the definition , the family of all fully invariant graded submodule of graded module M is non-empty. If $M = R$ then a ideal of R is fully invariant.

Let I, J graded ideals of S dan X a graded submodule of M, then we define $IJ = \{\sum_{1\le i\le n} x_i y_i | x_i \in \cup_{k\in G} I_k,\ y_i \in \cup_{k\in G} J_k, n \in \mathbb{N}\}$, and $I(X) = \sum_{f\in \cup_{k\in G} I_k} f(X)$. For any right graded R −module M and any right graded ideal I of graded ring R, the set MI is a fully invariant graded submodule of M.

Definition 2. 1. *Let M be a graded right R −module and X a fully invariant proper graded submodule of M. Then X is called a graded N −prime submodule of M if for any ideal I of S and any fully invariant graded submodule U of M, $I(U) \subset X$ implies $I(M) \subset X$ or $U \subset X$.*

Especially, if we take $M = R$ −module R, a graded ideal P of R is a graded prime ideal if for any graded ideals I, J of R with $IJ \subset P$ implies $I \subset P$ or $J \subset P$.

If R is a graded ring and M is a graded R −module , then R_e is subring of R and M_g is R_e − module. From this fact, we have the following definition.

Definition 2.2. *Let M be a graded right R −module , X a fully invariant proper graded submodule of M and $g \in G$. A fully invariant graded submodule X_g of R_e −module M_g is called N_g −prime submodule if for any ideal I_g of R_e and for any a fully invariant submodule U_g of M_g with $I_g(U_g) \subset X_g$ implies $I_g(M_g) \subset X_g$ or $U_g \subset X_g$.*

Relation between graded N −prime submodule and its homogeneous component is said the following lemma.

Proposisi 2.3. *Let R be a graded ring , M a graded R −module and X a fully invariant graded submodule of M. If X is a graded N −prime submodule of M then X_g is N_g −prime submodule of R_e −module M_g for every $g \in G$.*

Proof.
Let X be a graded N −prime submodule of graded module M. Take any ideal I_g of S_g and any a fully invariant submodule U_g of M_g such that $I_g(U_g) \subset X_g$. Because of $X_g \subset X$, then $I_g(U_g) \subset X_g \subset X$. From fact that X is a graded N −prime submodule, we have $I_g(M_g) \subset X$ or $U_g \subset X$. If $I_g(M_g) \subset X$, then $I_g(M_g) \subset X_g$ because X is graded submodule. If $U_g \subset X$ then $U_g \subset X_g$ because $U_g \subset M_g$ and $X_g = X \cap M_g$. Proving that $\mathrm{X_g}$ is $\mathrm{N_g}$ −prima submodule of $\mathrm{R_e}$ − modul $\mathrm{M_g}$ for every $\mathrm{g} \in \mathrm{G}$.
The following theorem gives some characterization of graded N −prime submodule. The proof the following theorem similar with theorem 1.2 in [7].

Theorem 2.4. *Let M be a graded R −module and X a proper fully invariant graded submodul of M. Then the following conditions are equivalent :*
(1). *X is a graded N −prime submodule of M*
(2).*For any a graded right ideal I of S, any graded submodule U of M if $I(U) \subset X$ then either $I(M) \subset X$ or $U \subset X$*
(3).*For any $\varphi \in h(S)$ and fully invariant graded submodule U of M, if $\varphi(U) \subset X$ then either $\varphi(M) \subset X$ or $U \subset X$*

Example 2.5. Let $G = \mathbb{Z}$, $R = \mathbb{Z} = R_0$ and $R_g = 0$ for $0 \neq g \in \mathbb{Z}$. R is graded ring . Let $M = \mathbb{Z}_4 \times \mathbb{Z}_4$ be a R −module. R −module Module M is graded module with $M_0 = \mathbb{Z}_4 \times \{0\}$, $M_1 = \{0\} \times \mathbb{Z}_4$ and $M_g = 0$ for $0{,}1 \neq g \in G$. Let $X = 2\mathbb{Z} \times \{0\}$ be a graded submodule , This submodule is graded N −prima submodule.

Lemma 2.6. *Let M be a graded a right R −module and $S = END_R(M)$. Suppose that X is a fully invariant graded submodule of M. Then the set $I_X = \{f \in S | f(M) \subset X\}$ is a graded ideal of S.*

Proof.
Take any $\varphi \in S$ and $\in I_X$. Then $\varphi f(M) \subset \varphi(X) \subset X$ and $f\varphi(M) \subset f(M) \subset X$. So φf , $f\varphi$ in I_X . It is a clear that $(I_X, +)$ is an abelian group. Proving that I_X is a ideal of S. Furthermore, we will prove that I_X is graded ideal of S, i.e $I_X = \bigoplus_{g\in G}(I_X \cap S_g)$ for every $g \in G$. For every $g \in G$, $I_X \cap S_g \subset I_X$, so we obtain $\bigoplus_{g\in G}(I_X \cap S_g) \subset I_X$. Take any $\in I_X$, Then $f = \sum_{g\in G} f_g$ and $f(M) = (\sum_{g\in G} f_g)(M) \subset X$, we will prove that $f \in \bigoplus_{g\in G}(I_X \cap S_g)$. Clear that $f_g \in S_g$, so we have to prove that $f_g \in I_X$ for every $g \in G$. From f_g is homogeneous component of f. Then $f_g \in S$ and $f_g(M) \subset f(M) \subset X$, so $f_g \in I_X$.

Theorem 2.7. *Let M be a graded right R −module, $S = END_R(M)$ and X a fully invariant garded submodule of M. If X is a graded N −prime submodule then I_X is a graded prime ideal of S*

Proof.
Let K, L be graded ideal of I_X such that $KL \subset I_X$. Then $KL(M) \subset I_X(M) \subset X$. If assume that $K \not\subset I_X$, than $K(M) \not\subset X$ and submodule X is graded N −prime submodule then $L(M) \subset X$, so we obtain $L \subset I_X$. Proving that I_X is a graded

prime ideal of S .

Let I be a graded ideal of S and M is a graded $R-$ modul. Let $I(M) = \sum_{f \in I} f(M)$ and $I_X = \{f \in S | f(M) \subset X\}$. We have The following proposition.

Proposition 2.8. *Let M be a graded $R-$module , X is a fully invariant graded submodule of M and I is a graded ideal of S. The set $I(M) = \sum_{f \in I} f(M) \subset X$ if only if $I \subset I_X = \{f \in S | f(M) \subset X\}$.*

Proof.
Take any $f \in I, f(M) \in I(M)$ and $I(M) \subset X$, we obtain $f(M) \in X$. From $I \subset S$, then $f \in S$, and we have $f \in I_X$. Conversely, let $I \subset I_X$ and the set $I(M) = \sum_{f \in I} f(M)$. From $I \subset I_X$, we have $\sum_{f \in I} f(M) \subset \sum_{f \in I_X} f(M) \subset X$

Teorema 2.9. *Let M be a graded $R-$modul , X is a fully invariant graded submodule of M .Then X is a graded $N-$prime submodule if only if for any graded ideal I of S and any fully invariant graded submodule U of M such that $I(U) \subset X$ implies $I \subset I_X$ or $U \subset X$.*

Proof.
Use definition 2.2 and proposition 2.8 .

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References

[1] Atani,S.E, 2006, On Graded Prime Submodules, *Chiang Mai J. Sci, 33(1),pp: 3-7.*

[2] Atani S.E, 2011, Notes the graded primes submodule, *International journal of Algebra*

[3] Bland,P.E, 2011, *Ring and Their Module,* Walter de Gruyter & Co.KG.Berlin/New York.

[4] Dauns, J, 1978, Prime Modules, *Journal fur die reine angewandte mathematic,* 298, pp:156-181.

[5] Natasescu,C and Oystaeyen V,F, 1982, *Graded Ring Theory,* North –Holland Pubishing Company.

[6] Oral, K.H, Tekir U, and Agargun,A.G, 2011, On graded prime and primary submodules, *Turk J Math, 35,pp:159-167.*

[7] Sanh, N.V, 2010, Primeness In Module Category, *Asian-European Journal of Mathematics.Vol 3, No 1, pp:145-154*

REGRESSION MODEL FOR SURFACE ENERGY MINIMIZATION BASED ON CHARACTERIZATION OF FRACTIONAL DERIVATIVE ORDER

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Abstract. If the midpoint of a rectangular elastic surface is pressed from the bottom to the top, then at all points on the surface will formed a potential energy that leads down. This energy will depend on the surface elasticity and the magnitude of the pressure. Assuming that the surface that occurred after pressed will be a function of two variables form a double sine series, and the energy that formed is an integral of the square of the Laplace operator that expanded into a fractional order, this paper will present a discussion about how to make a model of energy minimization problems that occurs, as well as characterizes the fractional derivatives order based on the surface elasticity.

Key words and Phrases : energy, elasticity, fractional, modeling, double sine.

1. Introduction

Suppose given a surface in form of a rectangular-shaped elastic plate. Furthermore we press under the object right in the middle of the surface. Obviously, at any point on the surface will occur opposite energy downward. This energy will depend on two things which are surface elasticity and the magnitude of surface tension. Similarly, if the surface is pressed on two or three different points, assuming the pressure exerted on the two or three points are equal, the energy will still depend on two things above.

Mathematically, if the square-shaped elastic surface is placed on the lines $x = 0$, $x = a$, $y = 0$, and $y = b$, and then pressed from under the surface at the center point, then it will look like in Figure-1 below.

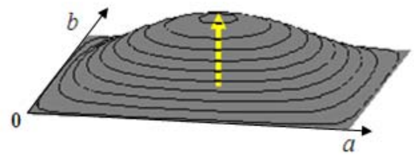


Figure-1

According Langhar [6], the pressed surface could be represented by the following double sine series function

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\, sin\frac{m\pi x}{a}\, sin\frac{n\pi y}{b} \tag{1}$$

that satisfies the boundary condition $w = 0$, $w_{xx} = 0$, and $w_{yy} = 0$ for $x = 0$ or $x = a$, and for $y = 0$ or $y = b$.

The tension energy on the downward-pointing plate surface is as follows:

$$E = \frac{1}{2}D\int_0^a\int_0^b \left(w_{xx} + w_{yy}\right)^2 dx\, dy \;=\; \frac{1}{2}D\int_0^a\int_0^b (\Delta w)^2\, dx\, dy \tag{2}$$

with D represents plate surface elasticity, that according to (1) obtained

$$E \;=\; \frac{1}{8}\,\pi^4 abD \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 a_{mn}{}^2\ . \tag{3}$$

2. Previous Research

In a previous research, Rusyaman *et al* [2] and H.Gunawan *et al* [4], have guaranteed the existence of function as interpolation results passing points $(x_i\ ,\ y_j\ ,\ c_{ij})$ shaped surface which is described as double sine series

$$u(x,y) := \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[a_{mn}\ \sin m\pi x\ \ \sin n\pi y\right] \tag{4}$$

with $0 \le y \le 1$ and $0 \le x \le 1$,and minimize the energy in form of

$$E_\beta(u) := \int_0^1\int_0^1 \left|(-\Delta)^{\beta/2}\, u\right|^2 dx\, dy\ . \tag{5}$$

where $\Delta = \dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}$ is Laplace operator, and β is the fractional-order of derivative.

With $u(x,y)$ as in (4), then the energy $E_\beta(u)$ will be as follow:

$$E_\beta(u) = \frac{1}{4}\pi^{2\beta}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}{}^2\left(m^2+n^2\right)^\beta \qquad (6)$$

where a function $u(x,y)$ exists, unique and continuous if and only if $\beta > 1$. [4]

The existence and uniqueness of the solution of this problem follows from the best approximation theory in Hilbert spaces [1] [3]. In this study, to obtain the solution of this problem, particularly in determining the value of the minimum energy, carried through the stages: determining initial function $u_0(x , y)$, determining and orthogonalizing basis by use the Gram Schmidt method, and the last is determining minimum energy value. Technically, this process uses iterative system with the help of software until obtain minimum energy value.

3. The Main Problem

Since the function in (1) is equal to the function (4) for $a = b = 1$, and the energy in (3) differ only in the constants in (6), the results in [1], [4] and [5] ie surface shape interpolation that minimizes energy, can be adopted as a surface that is pressed from below as in [6].

Due to determination of the minimum energy value require a long process, which must use the software through hundreds and even thousands of iterations, then in this paper it will be prepared some regression models which show the minimum energy as a function of β that can be identified with the surface elasticity, and C which states the amount of pressure on the surface.

By regression, the minimum energy value can be predicted by substituting the value of β and C into the model [5], as was done by Roberto Z Freire *et al* [7], in Development of Regression Equation for Predicting Energy and Hygrothermal Performance of Buildings.

With the existence of this regression model, the software will no longer be needed to determine the amount value of the energy, that we could simply use the model for easier and faster way.

The discussion of the problems is divided into three sections as follows :

1. Pressing from under a surface at a single point, which is right in the center (1/2 , 1/2 , 1) , as far as one unit , so that the energy only depends on β.

2. Pressing from under a surface at two different points are (1/4 , 1/2 , C) and (3/4 , 1/2 , C), as far as C units so that the energy depends on β and C.

3. Pressing from under a surface at three different points are (1/4 , 1/4 , C), (1/3 , 2/3 , C) and (3/4 , 1/2 , C) as far as C unit. Even in this case, the energy will depends on β and C

4. Main Results

4.1 Problem-1: Surface pressed at one point

In the first problem, given a surface in form of a rectangular-shaped elastic plate placed on lines $x = 0$, $x = 1$, $y = 0$, and $y = 1$. Furthermore midpoint of the plate is pressed from the bottom to the top as far as $C = 1$ unit as seem in Figure-1 above, so that the maximum point is in the middle of the field, with the exact coordinates (1/2 , 1/2 , 1).

The Surface formed is assumed as a function as in (4) and the value of the surface energy as in (5) and (6). The value of the energy E_β will depend on the value of β. Without losing the generality by eliminating constant $\pi^{2\beta}$ from the equation (6), by using the calculations have been described in [2] and [4] and using software until 1225th iteration, it obatined data of twenty minimum energy value are generated from twenty values of β with the range between 1.00 and 2.00 . Scatter plot diagram is shown in Figure-2 below, where the red line shows the least squares line, while the green line shows the linear regression line.

Data analisys results showed that the correlation between β and E_β is positive value, amounting to 0.9939006. This means that there is mutual influence which is proportional because its value is close to 1 , ie, the greater the value of β , the greater the value of E_β.

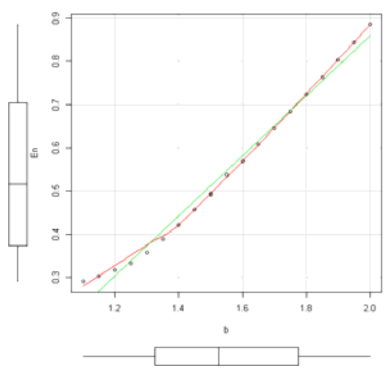


Figure-2

With a residual standard error of 0.02163, and freedom degrees of 18, as well as multiple R-squared amounting to 0.9878, then resulting intercept value = –0.52547 and β coefficient = 0.69240.

Thus, the linear regression model is:

$$E_\beta = -0.52547 + 0.69240\,\beta \;.$$

with error square of the model-1 is 0.00842482.

4.2 Problem-2: Surface pressed at two points

In the second problem, given a surface in form of a rectangular-shaped elastic plate placed on the lines $x = 0$, $x = 1$, $y = 0$, and $y = 1$, as shown in Figure-3. Then the plate is pressed from te bottom to the top on two distinct points, i.e (1/4, ½) and (3/4, ½) as far as C unit as seem in Figure-4 below. So that the two maximum points have coordinates (1/4 , ½ , C) and (3/4, 1/2, C).

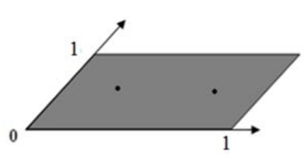


Figure-3

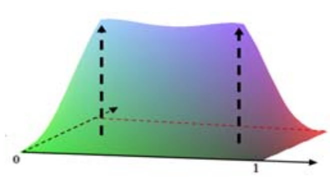


Figure-4

The value of the energy E_β will depends on the value of β and C. Without losing the generality by eliminating constant $\pi^{2\beta}$ from the equation (6), by using the calculations have been described in [2] and [4] and using software until 1225th iterations, it obatined data of twenty-five minimum energy value generated from twenty-five value of β with the range between 1.0 until 2.2 and also twenty-five value of C with distribution between 0.1 until 1.0.

Data analisys results showed that the correlation between β and E_β is positive value amounting to 0.1778537. This means that proportionally less influence because its value is relatively small, while correlation between C and E_β is better since the value is close to 1, i.e 0.8507899. Since the correlation value is positive, then both β and C are proportional to E_β . Therefore, the greater the value of β and C, the greater the value of E_β .

With the residual standard error of 0.2518, and the freedom degrees of 22, as well as multiple R-squared of 0.8528 resulting intercept value = –1.5651, coefficient of β = 0.6978, and the coefficient of C = 1.8696 .

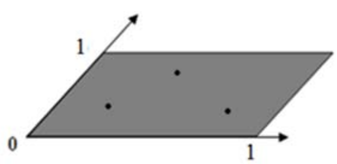
Thus the linear regression model is :

$$E_\beta = -1.5651 + 0.6978\,\beta + 1.8696\,C \quad .$$

with error square of the model-2 is 1.39497136

4.3 Problem-3: Surface pressed at three points

On the third problem, given a surface in form of a rectangular-shaped elastic plate placed on the lines x = 0, x = 1, y = 0, and y = 1, as shown in Figure-5. Then the plate is pressed from the bottom to the top on three distinct points, i.e (1/4 , 1/4), (1/3 , 2/3) and (3/4 , 1/2) as far as C-units as in Figure-6 below, so that the three maximum points have coordinates (1/4 , 1/4 , C), (1/3 , 2/3 , C) and (3/4 , 1/2 , C).



Figurer-5

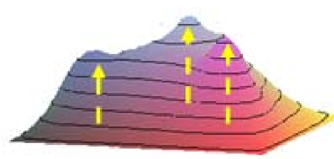


Figure-6

Same with the two-points problem, the value of the energy E_β will depends on the value of β and C. Without losing the generality by eliminating constant $\pi^{2\beta}$ from the equation (6) , by using the calculations have been described in [2] and [4] and using software until 1225th iteration, it obatined data of twenty-five minimum energy value generated from twenty-five value of β with the range between 1.0 until 2.2 and also twenty-five value of C with range between 0.1 until 1.0.

Data analisys results showed that the correlation between β and E_β is positive numbers amounting to 0.2774373, means proportionally less influence because its value is relatively small. While correlation between C and E_β is better since the value is close to 1, i.e 0.8131555. Since the correlation value is positive, then both β and C are proportional to E_β . Therefore, the greater the value of β and C, the greater the value of E_β.

With a residual standard error of 0.2104 , and the freedom degrees of 22, as well as multiple R - squared of 0.8578 resulting intercept value = –1.5348, coefficient β = 0.7362, and the coefficient C = 1.6467 .

Thus the linear regression model is :

$$E_\beta = -1.5348 + 0.7362\,\beta + 1.6467\,C \quad .$$

with error square of the model-3 is 0.9742403.

5. Conclusion

The data processing and analysis of model resulted in a conclusion that the value of the fractional-order derivative β, can be identified with surface elasticity as denoted by D in equation (2) and (3). Furthermore, the value of β and C ie the amount of pressure from buttom is directly proportional to the amount of the minimum energy E_β which means the more rigid material surface, and the greater the pressure exerted, then the greater the energy value.

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References

[1] Atkinson.K & Han.W, 2001, Theorical Numerical Analysis A Functional Analysis Framework, Springer Verlag, New York,.

[2] E. Rusyaman, H. Gunawan, A.K. Supriatna, R.E. Siregar, 2009, A 2-D Interpolation Method That Minimizes An Energy Integral, Presented at IICMA in Yogyakarta,.

[3] H. Gunawan, F. Pranolo, E. Rusyaman, 2007 , An Interpolation Method That Minimizes an Energy Integral of Fractional Order. Singapore: Proceeding of Asian Symposium of Computer Mathematics.

[4] H. Gunawan, E. Rusyaman & L. Ambarwati, 2011, Surfaces with Prescribed Nodes and Minimum Energy Integral of Fractional Order ITB J.Sci., Vol. 43 A, No. 3, page 209-222.

[5] Hogg and Craig,1995 , Introduction to Mathematical Statistics, Prentice-Hall, New Jersey. .

[6] Langhaar.H.L, 1962, Energy Methods in Applied Mechanics, John Wiley & Sons, New York.

[7] Roberto Z Freire, Gustavo H. C Oliveira, and Nathan Mendes, 2008, Development of regression Equation for Predicting Energy and Hygrothermal Performance of Buildings, Energy and Building 40 , page 810-820, www.sciencedirect.com.

THE HENSTOCK-STIELTJES INTEGRA IN R^n

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Abstract. Lim, et.al. [4] studied the Henstock-Stieltjes integral on the real line and proved some of its properties. In this paper, we discuss the Henstock-Stieltjes integral in the $n-$ dimensional Euclidean space ($\boldsymbol{R^n}$). The main purpose of this article is to define the Henstock-Stieltjes integral in $\boldsymbol{R^n}$ and develop some of its characteristics. At first, we define a bounded variation function on a cell $E \subset \boldsymbol{R^n}$. We use the equivalency between the interval function and point function on a cell $E \subset \boldsymbol{R^n}$ to construct the definition. Some characterizations of bounded variation function defined on a cell $E \subset \boldsymbol{R^n}$ will be developed. The results will be used to give the sufficient condition for existence of the Henstock-Stieltjes integral in $\boldsymbol{R^n}$ and to investigate the properties of the constructed integral. Some convergence theorems will be stated at the end of the paper

Keywords and Phrases: Henstock-Stieltjes integral, bounded variation function, convergence theorems.

1. Introduction

Integral theory is a branch of mathematics that is deductive analysis and it is still growing and developing both theoretical and applicative. In its development, integral theory emerges through a descriptive and constructive definition [2]. At first Henstock-Stieltjes integral applies only to functions with monotonous integrator. Lim in 1998 studied the Henstock-Stieltjes integral on the real line and gave some sufficience conditions for a function f to be Henstock-Stieltjes on $[a,b]$ with respect to an increasing function g on $[a,b]$ [4]. Furthermore the emergence of the concept bounded variation functions give an opportunities to analyze the existence of the Henstock-Stieltjes integral in a larger class of functions. A real function on $[a,b]$ has bounded variation if its total variation is finite. A function of bounded variation on $[a,b]$ can be expressed as a difference of two increasing functions [3]. From this result, the Henstock-Stieltjes integral with respect to the integrator of monotone function can be generalized with respect to the integrator of bounded variation function.

A more generalized concept of space in $n-$ dimensional Euclidean space $(\mathbf{R}^n)$ inspires researchers including [2] and [5] which give various definitions and basic properties of multidimensional integrations. Indrati [1] generalizes the Henstock-Kurzweil integral in the $n-$dimensional Euclidean space. The opportunities for developing Henstock-Stieltjes integral with respect to the bounded variation function integrator in $\mathbf{R}^n$ are still widely open. This is inspired by the success of [1] who developed the Henstock-Kurzweil integral in the $n-$ dimensional Euclidean space. The main purpose of this article is to define and develop Henstock-Stieltjes integral in $\mathbf{R}^n$. First, we define a bounded variation function on a cell $E \subset \mathbf{R}^n$. Next, we use the equivalency between the interval function and point function on a cell $E \subset \mathbf{R}^n$ to construct the definition. Some characterizations of bounded variation function defined on a cell $E \subset \mathbf{R}^n$ will be developed. Based on those characteristics, we generalized the Henstock-Stieltjes integral with respect to the integrator of bounded variation function on the cell $E \subset \mathbf{R}^n$. Then, we investigate the properties of the constructed integral and give the sufficient condition for existence of the Henstock-Stieltjes integral in $\mathbf{R}^n$. Some convergence theorems will be stated at the end of the paper.

Before discussing the main results of the research, we begin with some terminology notations. The scope of this paper is the $n-$ dimensional Euclidean space $(\mathbf{R}^n)$, with $\mathbf{R}^n = \{\bar{x} = (x_1, x_2, x_3, ..., x_n); x_i \in \mathbf{R}, \text{ and } 1 \le i \le n\}$. For $\bar{x} = (x_1, x_2, ..., x_n) \in \mathbf{R}^n, \|\bar{x}\| = \max|x_i| : 1 \le i \le n$ and the distance between two points $\bar{x}, \bar{y} \in \mathbf{R}^n$ is defined as follows:

$$d(\bar{x}, \bar{y}) = \|\bar{x} - \bar{y}\|_\infty = \max_{1 \le i \le n}|x_i - y_i| .$$

Furthermore, we also have four types of interval in $\mathbf{R}^n$. For $\bar{a}, \bar{b} \in \mathbf{R}^n$ we have

(i). $[\bar{a}, \bar{b}] = \{\bar{x} = (x_1, x_2, ..., x_n) : a_i \le x_i \le b_i, i = 1,2,3,...,n\}$

(ii). $(\bar{a}, \bar{b}] = \{\bar{x} = (x_1, x_2, ..., x_n) : a_i < x_i \le b_i, i = 1,2,3,...,n\}$

(iii). $[\bar{a}, \bar{b}) = \{\bar{x} = (x_1, x_2, ..., x_n) : a_i \le x_i < b_i, i = 1,2,3,...,n\}$

(iv). $(\bar{a}, \bar{b}) = \{\bar{x} = (x_1, x_2, ..., x_n) : a_i \le x_i \le b_i, i = 1,2,3,...,n\}$

In this paper, a cell $E \subset \mathbf{R}^n$ stands for a non-degenerate closed and bounded interval $[\bar{a}, \bar{b}]$, where $\bar{a} < \bar{b}$. Its volume will be represented by $|E|$. Then if E a cell, $\bar{x} \in E$, and δ is a positive function on E, we defined an open ball centered at $\bar{x} \in E$ with radius $\delta(\bar{x})$, represented by $B(\bar{x}, \delta(\bar{x}))$, is defined as $B(\bar{x}, \delta(\bar{x})) = \{\bar{y} : \|\bar{x} - \bar{y}\| < \delta(\bar{x})\}$.

Next, we will discuss about the $\delta-$fine partition and the $\delta-$fine interval. A collection of cells $\{I_i : i = 1,2,3,..., p\}$ is called non-overlapping if $I_i^{\,o} \cap I_j^{\,o} = \phi$, for $i \ne j$. A collection of a finite non-overlapping cell $\mathfrak{D} = \{I\} = \{I_1, I_2, ..., I_p\}$, with $\bigcup_{I \in \mathfrak{D}} I = \bigcup_{i=1}^{p} I_i = E$ is called *a partition* of E. By taking a positive function δ defined on a cell E, we get a collections of the pairs of a finite point-cell $\mathfrak{D} = \{(I, \bar{x})\} = \{(I_i, \bar{x}_i); i = 1,2,3,..., p\}$ with $\{\bar{x}_1, \bar{x}_2, \bar{x}_3, ..., \bar{x}_p\} \subset E$ and

$I_i^o \cap I_j^o = \phi$ for $i \neq j$, and its called *a δ-fine partition* on a cell E if $I_i \subset B(\bar{x}_i, \delta(\bar{x}_i)) \subset E$, for $i = 1,2,\ldots,p$ and $E = \bigcup_{i=1}^{p} I_i$. If $x_i \in I_i$, for $i = 1,2,\ldots,p$ then $\mathfrak{D} = \{(I,\bar{x})\} = \{(I_i,\bar{x}_i); i = 1,2,3,\ldots,p\}$ is called *Perron δ-fine partition* on a cell E. The point $\bar{x}$ with $(I,\bar{x}) \in \mathfrak{D}$ is called *associated point* and $(I,\bar{x}) \in \mathfrak{D}$ is called *δ-fine interval* with associated point $\bar{x}$.

2. Bounded Variation Function in The $n-$ Dimensional Euclidean Space

We shall now discuss the concept of functions of bounded variation in the $n-$ dimensional Euclidean space. A function of bounded variation is a real-valued function whose total variation is bounded. In 2000, Lee and Vyborny gave relation between monotonic function and bounded variation function. A function of bounded variation can be expressed as a difference of two increasing functions [3]. The generalization of the definition of bounded variation function in multidimensional space is started by defining interval function and point function on a cell $E \subset \mathbf{R}^n$.

A function $F : \mathcal{J}(E) \to \mathbf{R}$ is called real valued interval function. A function F is said to be an additive interval function if for every $I_1, I_2 \in \mathcal{J}(E)$ with $I_1^o \cap I_2^o = \phi$ and $I_1 \cup I_2$ interval then $F(I_1 \cup I_2) = F(I_1) + F(I_2)$. An interval function can be constructed by the point function and otherwise point function can be constructed by the intervals function.

Definition 2.1. *Let $E = [\bar{a},\bar{b}] \subset \mathbf{R}^n$ be a cell, $\mathcal{J}(E)$ be a collection of all intervals subset E and $F : \mathcal{J}(E) \to \mathbf{R}$ is an interval function. Based on the interval function F, we may define point function $f : E \to \mathbf{R}$. If $\bar{x} \in E$, the point function f at $\bar{x}$, written by $f(\bar{x})$ is defined as follows:*

$$f(\bar{x}) = \begin{cases} 0, & \text{if } \bar{x} \text{ with } x_i = a_i \text{ for at least one } i \\ F([\bar{a},\bar{x}]), & \text{for } \bar{x} \text{ otherwise}. \end{cases}$$

Opposite case is given in the following definition. Given a point function f defined on E, we may defined a corresponding interval function F as follows.

Definition 2.2. *Let $E = [\bar{a},\bar{b}] \subset \mathbf{R}^n$ be a cell and $f : E \to \mathbf{R}$ is a point function. Based on the interval function f, with the value of the function f at the point $\bar{x} \in E$ written by $f(\bar{x})$. We may define interval function $F : \mathcal{J}(E) \to \mathbf{R}$ as follows. If $I = [\bar{c},\bar{d}] \subset E$ with $\bar{c} = (c_1,c_2,c_3,c_4,\ldots,c_n)$, $\bar{d} = (d_1,d_2,d_3,d_4,\ldots,d_n)$ and $\bar{\gamma} = (\gamma_1,\gamma_2,\gamma_3,\gamma_4,\ldots\gamma_n) \in I$ where $\gamma_i \in \{c_i,d_i\}$ for $i = 1,2,\ldots,n$, and $n(\bar{\gamma})$ denotes the number of terms in $\bar{\gamma}$ for which $\gamma_i = c_i$ for any i with $i = 1,2,\ldots,n$, then the value function F at I, written by $F(I)$, denotes $F(I) = \sum (-1)^{n(\bar{\gamma})} f(\bar{\gamma})$. Where the summation is over all vertices $\bar{\gamma}$ and in this* *case $\bar{\gamma}$ stated the corner points of the interval I.*

Furthermore, we have monotonically properties between a corresponding interval function and point function. The point function has monotonically properties if only if the interval function has monotonically properties. From the equivalency between the interval function and point function on a cell $E \subset \mathbf{R}^n$, we construct the definition of bounded variation function in the $n-$ Dimensional Euclidean Space. Some characterizations of bounded variation function defined on a cell $E \subset \mathbf{R}^n$ will be developed.

Definition 2.3. Let $E=[\bar{a},\bar{b}]\subset\mathbf{R}^n$ *be a cell and* $F:\mathfrak{I}(E)\to\mathbf{R}$ *interval function. The variation of* F *on a cell* E*, written by* $V_F(E)$ *or* $V(F;E)$*, is define as follows*

$$V_F(E)=V(F;E)=\sup\left\{\sum_{I\in\mathfrak{D}}\left|F(I)\right|\right\},$$

with supremum is over all $\mathfrak{D}$ *partition in* E. *If* $V_F(E)<\infty$ *then* F *is said to be of bounded variation on* E.

Let $BV(E)$ stand for a collection of all bounded variation function on a cell $E\subset\mathbf{R}^n$ to a real number $\mathbf{R}$. If F is bounded variation function on a cell E then we will be represented by $F\in BV(E)$. Furthermore, if an additive interval function F is bounded variation function on a cell E then F is bounded.

Theorem 2.4. *Let* $E=[\bar{a},\bar{b}]\subset\mathbf{R}^n$ *be a cell,* $X\subseteq E$*,* $F,G:\mathfrak{I}(E)\to\mathbf{R}$ *are additive interval function and* c *is a real number. If* $F,G\in BV(X)$ *then* $F+G$*,* cF*,* FG*,* $\max\{F,G\}$*,* $\min\{F,G\}\in BV(X)$ *with* $(F+G)(I)=F(I)+G(I)$*,* $(cF)(I)=cF(I)$ *and* $(FG)(I)=F(I)G(I)$ *for every* $I\in\mathfrak{I}(E)$ *with the starting and end point of* I *belong to* X.

Based on Theorem 2.4. we have that $BV(E)$ is a linear space, because if $F,G\in BV(X)$ then $F+G\in BV(X)$, and $cF\in BV(X)$ for every $c\in\mathbf{R}$. Next we will discuss properties of bounded variation interval functions associated with the domain.

Theorem 2.5. *Let* $E=[\bar{a},\bar{b}]\subset\mathbf{R}^n$ *be a cell. An additive interval function* $F:\mathfrak{I}(E)\to\mathbf{R}$ *and* $I_1,I_2\subseteq E$ *two non-overlaping cell. For each* $X_1\subseteq I_1$ *dan* $X_2\subseteq I_2$ *if* $F\in BV(X_1)$ *and* $F\in BV(X_2)$ *then* $F\in BV(X_1\cup X_2)$.

From Theorem 2.5., we have the corollary 2.6.

Corollary 2.6. *Let* $E=[\bar{a},\bar{b}]\subset\mathbf{R}^n$ *be a cell, interval function* $F:\mathfrak{I}(E)\to\mathbf{R}$ *and* $X_1,X_2\subseteq E$. *If* $F\in BV(X_2)$ *and* $X_1\subseteq X_2$ *then* $F\in BV(X_1)$ *and then* $V_F(X_1)\le V_F(X_2)$.

That means $V_F(.)$ is increasing interval function. Furthermore, based on Corollary 2.6. we have the Corollary 2.7.

Corollary 2.7. *Let $E=[\overline{a},\overline{b}]\subset \mathbf{R}^n$ be a cell and F is an additive interval function. A function $F \in BV(E)$ if and only if it is the difference of two increasing functions on E.*

In the other word, if a function F on a cell E has a bounded variation then it is can defined with the difference of two increasing functions on E, and if F is increasing function on a cell E then $F \in BV(E)$. Next, we will discuss the bounded variation properties in the point function and the associated interval function.

Definition 2.8. *Let $E=[\overline{a},\overline{b}]\subset \mathbf{R}^n$ be a cell and $f: E \to \mathbf{R}$ is a point function. The oscillation function f, written by $\omega(f,\cdot)$, with $\omega(f,\cdot): \mathfrak{I}(E) \to \mathbf{R}^*$ defined by $\omega(f,I)=\sup\limits_{\overline{x},\overline{y}\in I}\left|f(\overline{x})-f(\overline{y})\right|$, for each $I \in \mathfrak{I}(E)$.*

We defined the bounded variation of point function on a cell E as in Definition 2.9.

Definition 2.9. *Let $E=[\overline{a},\overline{b}]\subset \mathbf{R}^n$ be a cell and $f: E \to \mathbf{R}$ is a point function. A point function f is said to be of bounded variation on E, if there exist a positif number $M \geq 0$ such as for every $\mathfrak{D}=\{I_i; i=1,2,3,..m\}$ partition on a cell E we have*

$$(\mathfrak{D})\sum \omega(f,I)=\sum_{i=1}^{m}\omega(f,I_i)\leq M .$$

The number of $V_f(E)=V(f,E)=\sup(\mathfrak{D})\sum \omega(f,I)$ is said to be a variation of point function f on a cell E, with the supremum is over all $\mathfrak{D}=\{I_i; i=1,2,3,..m\}$ partition in E.

Furthermore, if a point function f is bounded variation function on a cell E then f is bounded. Beside that $BV(E)$ is a linear space. In addition the following theorem applies.

Theorem 2.10. *Let $E=[\overline{a},\overline{b}]\subset \mathbf{R}^n$ be a cell, $f: E \to \mathbf{R}$ is a point function and $F: \mathfrak{I}(E) \to \mathbf{R}$ is an additive interval function that associated with point function f. A function $f \in BV(E)$ if and only if $F \in BV(E)$.*

Based on the theorem is concluded that the bounded variation properties is maintained for the point function and intervals function are interlinked with each other. Because in the next chapter we will discuss the generalization of integral with respect to the bounded variation function G then we need more general definition of the continuity, its called the continuity with respect to G.

Definition 2.11. *Let* $E=[\bar{a},\bar{b}]\subset \mathbf{R}^n$ *be a cell,* $F,G:\mathcal{I}(E)\to \mathbf{R}$ *are additive functions on* E *and* $G\in BV(E)$. *We have*

(i). *An interval function* F *is said to be continuous with respect to* G *at a cell* E *if for every* $\varepsilon>0$ *there is* $\eta>0$ *such that for any cells* $I\subseteq E$ *with* $|G(I)|<\eta$ *then* $|F(I)|<\varepsilon$.

(ii). *An interval function* F *is said to be continuous with respect to* G *at an open interval* $U\subseteq E$*, if for every* $\varepsilon>0$ *there is* $\eta>0$ *such that for any cells* $I\subseteq U$ *with* $|G(I)|<\eta$ *then* $|F(I)|<\varepsilon$.

Based on those definition, in the next section, we generalized the Henstock-Stieltjes integral with respect to the integrator of bounded variation function on cell $E\subset \mathbf{R}^n$.

3. The Henstock-Stieltjes Integral with Respect to Integrator of Bounded Variation Function in The $n-$ Dimensional Euclidean Space

The Henstock-Stieltjes integral in $\mathbf{R}^n$ is generalization of the Henstock-Stieltjes integral on the interval $[a,b]$ in the real number system $\mathbf{R}$. Bounded variation is important to the existence of Henstock–Stieltjes integral in real line. Beside that the Henstock-Stieltjes integral in $\mathbf{R}^n$ is also generalization of the Henstock integral in $\mathbf{R}^n$. The success of Indrati (2002), who developed the Henstock-Kurzweil integral in the $n-$ dimensional Euclidean space, is opening the opportunities for developing Henstock-Stieltjes integral with respect to the bounded variation function integrator in $\mathbf{R}^n$. The discussion will begin by establishing definitions and basic properties of Henstock-Stieltjes integral with respect to the integrator G on a cell $E\subset \mathbf{R}^n$.

Definisi 3.1. *Let* $E=[\bar{a},\bar{b}]\subset \mathbf{R}^n$ *be a cell, a function* $G:E\to \mathbf{R}$ *is a bounded variation function on a cell* E, δ *is a positif function on* E *and let* $\mathfrak{D}=\{(I,\bar{x})\}=\{(I_i,\bar{x}_i);i=1,2,3,\ldots,p\}$ *be a Perron* δ*-fine partition on a cell* E. *For any function* f *on* $\{x_1,x_2,x_3,\ldots,x_p\}$*, we set*

$$\sigma(f,\mathfrak{D},G)=(\mathfrak{D})\sum f(\bar{x})G(I)=\sum_{i=1}^{p} f(\bar{x}_i)G(I_i).$$

and call this number the G*-Stieltjes Sum of* f *associated with Perron* δ*-fine* *partition* $\mathfrak{D}$ *on a cell* E.

Now, we present the definition of the Henstock-Stieltjes integral by constructing the definition of the Henstock-Stieltjes integral with take on a bounded variation functions $G : E \to \mathbf{R}$ as an integrator of the integral.

Definition 3.2. *Let* $E = [\overline{a}, \overline{b}] \subset \mathbf{R}^n$ *be a cell and* $G : E \to \mathbf{R}$ *is a bounded variation function on a cell* E. *A function* $f : E \to \mathbf{R}$ *is said to be Henstock-Stieltjes integrable (HS- integrable) with respect to* G *on* E, *in short,* $f \in HS(G,E)$, *if there is a real number* A *such that for every* $\varepsilon > 0$ *there is a positive function* δ *on* E *such that for every Perron* δ*-fine partition* $\mathfrak{D} = \{(I, \overline{x})\} = \{(I_i, \overline{x}_i); i = 1,2,3,..., p\}$ *of* E, *we have*

$$\left|(\mathfrak{D})\sum f(\overline{x})G(I) - A\right|,$$

where $(\mathfrak{D})\sum f(\overline{x})G(I) = \sum_{i=1}^{p} f(\overline{x}_i)G(I_i)$.

The numbers A in Definition 3.2. is unique and called the value of Henstock-Stieltjes integral and will be written as $A = (HS)\int_E f dG$. If E is a degenerate interval, we defined $(HS)\int_E f dG = 0$. The $HS(G,E)$ in Definition 3.2. represent a collection of all Henstock-Stieltjes integrable function from a cell E to $\mathbf{R}$ with respect to a bounded variation function G on a cell $E \subset \mathbf{R}^n$. Furthermore, the function f is called ***integrand*** and the function G is called ***integrator***. If the integrator function G is an identity function $G(I) = I$ for every $I \in \mathcal{J}(E)$ then The Henstock-Stieltjes integral become the Henstock integral in $\mathbf{R}^n$. Thus the Henstock integral in $\mathbf{R}^n$ is one of the special cases of the Henstock-Stieltjes integral in $\mathbf{R}^n$. From the Definition 3.2., we have some basic properties of the Henstock-Stieltjes integral as follows.

Theorem 3.3. *Let* $E = [\overline{a}, \overline{b}] \subset \mathbf{R}^n$ *be a cell,* $f, f_1, f_2 : E \to \mathbf{R}$ *are functions on* E *and* $G : E \to \mathbf{R}$ *is a bounded variation function on a cell* E.

(i) *If* $f \in HS(G,E)$ *and* $\alpha \in \mathbf{R}$ *then* $\alpha f \in HS(G,E)$ *and* $f \in HS(\alpha G, E)$.
Moreover, we have
$$(HS)\int_E \alpha f dG = \alpha (HS)\int_E f dG = (HS)\int_E f d(\alpha G).$$

(ii) *If* $f_1, f_2 \in HS(G,E)$ *then* $f_1 + f_2 \in HS(G,E)$.
Moreover, we have
$$(HS)\int_E (f_1 + f_2) dG = (HS)\int_E f_1 dG + (HS)\int_E f_2 dG .$$

(iii) *If* $f \in HS(G,E)$ *and* $f \in HS(H,E)$ *then* $f \in HS(G + H, E)$.
Moreover, we have
$$(HS)\int_E f d(G + H) = (HS)\int_E f dG + (HS)\int_E f dH .$$

Based on the Theorem 3.3., we have that $HS(G,E)$ is a linear space. For the next discussion, we define the Cauchy criterion that provides an alternative to determine of Henstock-Stieltjes integrable function on a cell $E \subset \mathbf{R}^n$ without having to determine the value of the integral.

Theorem 3.4. (Cauchy's Criterion Theorem) *Let $E = [\bar{a}, \bar{b}] \subset \mathbf{R}^n$ be a cell and $G : E \to \mathbf{R}$ is a bounded variation function on a cell E. A function $f \in HS(G,E)$ if and only if for every $\varepsilon > 0$ there exists a positive function δ on E such that for every two Perron δ - fine partitions $\mathfrak{D}_1 = \{(I,\bar{x})\}$ dan $\mathfrak{D}_2 = \{(I,\bar{x})\}$ of E, we have*

$$\left|(\mathfrak{D}_1)\sum f(\bar{x})G(I) - (\mathfrak{D}_2)\sum f(\bar{x})G(I)\right| < \varepsilon .$$

PROOF. $(\Rightarrow)$ From $f \in HS(G,E)$, there *is a real number* $A = (HS)\int_E f dG$ *such that for every $\varepsilon > 0$ there is a positive function δ on E such that for every Perron δ-fine partition $\mathfrak{D}_1 = \{(I,\bar{x})\}$ of E and Perron δ-fine partition $\mathfrak{D}_2 = \{(I,\bar{x})\}$ of E, we have*

$$\left|(\mathfrak{D}_1)\sum f(\bar{x})G(I) - A\right| < \frac{\varepsilon}{2} \text{ and } \left|(\mathfrak{D}_2)\sum f(\bar{x})G(I) - A\right| < \frac{\varepsilon}{2} .$$

Then we have

$$\begin{aligned}\left|(\mathfrak{D}_1)\sum f(\bar{x})G(I) - (\mathfrak{D}_2)\sum f(\bar{x})G(I)\right| &= \left|(\mathfrak{D}_1)\sum f(\bar{x})G(I) - A + A - (\mathfrak{D}_2)\sum f(\bar{x})G(I)\right| \\ &\le \left|(\mathfrak{D}_1)\sum f(\bar{x})G(I) - A\right| + \left|A - (\mathfrak{D}_2)\sum f(\bar{x})G(I)\right| \\ &= \left|(\mathfrak{D}_1)\sum f(\bar{x})G(I) - A\right| + \left|(\mathfrak{D}_2)\sum f(\bar{x})G(I) - A\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .\end{aligned}$$

$(\Leftarrow)$ For every $\varepsilon = \frac{1}{n}$, for $n \in N$, there is a positive function δ_n^* on E such that for any two Perron δ_n^* - fine partitions $\mathfrak{D}_1^* = \{(I', \bar{x})\}$ and $\mathfrak{D}_2^* = \{(I'', \bar{x})\}$ of E, we have

$$\left|(\mathfrak{D}_1^*)\sum f(\bar{x})G(I') - (\mathfrak{D}_2^*)\sum f(\bar{x})G(I'')\right| < \frac{1}{n}.$$

For every $n \in N$, we define a positive function δ_n on E, where $\delta_n(\bar{x}) = \min\{\delta_1^*(\bar{x}), \delta_2^*(\bar{x}), \delta_3^*(\bar{x}), ..., \delta_n^*(\bar{x})\}$ for every $\bar{x} \in E$. From the definition we have that $\delta_n \le \delta_m$, for every $n \ge m$, so we have, every Perron δ_n - fine partition on E also Perron δ_m - fine partition on E, when $n \ge m$. For every n, we can choose only one Perron δ_n -fine partition $\mathfrak{D}_n = \{(I,\bar{x})\}$ on E then we construct $S_n = (\mathfrak{D}_n)\sum f(\bar{x})G(I)$. Moreover, we have $\{S_n\}$ the sequence in $\mathbf{R}$.

With Archimedian properties, that given any real number $\varepsilon > 0$ there exists natural number $n_o > \frac{1}{\varepsilon}$, then for every $m,n \geq n_o$, $n > m$ and by taking δ_n a positive function on E, we have

$$|S_n - S_m| = \left|(\mathfrak{D}_n)\sum f(\bar{x})G(I) - (\mathfrak{D}_m)\sum f(\bar{x})G(I)\right| < \frac{1}{m} \leq \frac{1}{n_o} < \varepsilon .$$

Thus obtained that $\{S_n\}$ is a Cauchy sequence in $\mathbf{R}$, then $\{S_n\}$ is convergence sequence. So, there exists a real number A as the limit of $\{S_n\}$. That means for every $\varepsilon > 0$ as the above, there exist natural number $n' > \frac{2}{\varepsilon}$, then for every $n \in \mathbf{N}$, $n > n'$ we have $|S_n - A| < \frac{\varepsilon}{2}$. Let $\delta = \delta_{n'}$ is a positive function on E, then for every Perron δ -fine Partition $\mathfrak{D} = \{(I, \bar{x})\}$, on E we have

$$\begin{aligned}\left|(\mathfrak{D})\sum f(\bar{x})G(I) - A\right| &= \left|(\mathfrak{D})\sum f(\bar{x})G(I) - S_{n'} + S_{n'} - A\right| \\ &\leq \left|(\mathfrak{D})\sum f(\bar{x})G(I) - S_{n'}\right| + |S_{n'} - A| \\ &< \frac{1}{n'} + \frac{\varepsilon}{2} \quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .\end{aligned}$$

So we can proved that A is the Henstock-Stieltjes integral value of f with respect to G on E. That means $f \in HS(G,E)$. □

Theorem 3.5. *Let $E = [\bar{a}, \bar{b}] \subset \mathbf{R}^n$ be a cell , $G : E \to \mathbf{R}$ is a bounded variation function on a cell E and E_1, E_2 be non-overlapping cell in $\mathbf{R}^n$ with $E = E_1 \cup E_2$. if $f \in HS(G,E_1)$ and $f \in HS(G,E_2)$ then $f \in HS(G,E)$. Moreover, we have*

$$(HS)\int_E f dG = (HS)\int_{E_1} f dG + (HS)\int_{E_2} f dG .$$

By Cauchy's Criterion and Theorem 3.5. we have Theorem 3.6. as follows.

Theorem 3.6. *Let $E = [\bar{a}, \bar{b}] \subset \mathbf{R}^n$ be a cell, $B \subseteq E$ and $G : E \to \mathbf{R}$ is a bounded variation function on a cell E. If $f \in HS(G,E)$ then $f \in HS(G,B)$ for every cell $B \subseteq E$.*

PROOF. Since $f \in HS(G,E)$, By Cauchy's Criterion, for every $\varepsilon > 0$, there exists a positive function δ on E such that for every two Perron δ - fine partitions $\mathfrak{D}^* = \{(I, \bar{x})\}$ and $\mathfrak{D}^{**} = \{(I', \bar{x})\}$ of E, we have

$$\left|(\mathfrak{D}^*)\sum f(\bar{x})G(I) - (\mathfrak{D}^{**})\sum f(\bar{x})G(I')\right| < \varepsilon .$$

If $B = E$ then we have the trivial proof. If $B \subset E$ then there exists collection of non–overlapping finite subcells E, $\mathscr{C} = \{C_1, C_2, C_3, ..., C_m\}$, such that

$cl(E-B)=\bigcup_{i=1}^{m} C_i$. By Cousin's lemma there exists Perron δ -fine partition $\mathfrak{D}_i$ on a cell C_i, for every $C_i \in \mathscr{C}$, $i=1,2,3,..,m$. Furthermore, if $\mathfrak{D}_B$ and $\mathfrak{P}_B$ are Perron δ -fine partition on a cell B then $\mathfrak{D}' = \mathfrak{D}_B \cup \left(\bigcup_{i=1}^{m}\mathfrak{D}_i\right)$ dan $\mathfrak{D}'' = \mathfrak{P}_B \cup \left(\bigcup_{i=1}^{m}\mathfrak{D}_i\right)$ also Perron δ -fine partition on E. Then we have

$$\left|(\mathfrak{D}_B)\sum f(\bar{x})G(I)-(\mathfrak{P}_B)\sum f(\bar{x})G(I)\right|$$
$$=\left|(\mathfrak{D}_B)\sum f(\bar{x})G(I)+\left(\bigcup_{i=1}^{m}\mathfrak{D}_i\right)\sum f(\bar{x})G(I)-\left(\bigcup_{i=1}^{m}\mathfrak{D}_i\right)\sum f(\bar{x})G(I)-(\mathfrak{P}_B)\sum f(\bar{x})G(I)\right|$$
$$=\left|\left((\mathfrak{D}_B)\sum f(\bar{x})G(I)+\left(\bigcup_{i=1}^{m}\mathfrak{D}_i\right)\sum f(\bar{x})G(I)\right)-\left(\left(\bigcup_{i=1}^{m}\mathfrak{D}_i\right)\sum f(\bar{x})G(I)+(\mathfrak{P}_B)\sum f(\bar{x})G(I)\right)\right|$$
$$=\left|\left(\mathfrak{D}_B\cup\left(\bigcup_{i=1}^{m}\mathfrak{D}_i\right)\right)\sum f(\bar{x})G(I)-\left(\mathfrak{P}_B\cup\left(\bigcup_{i=1}^{m}\mathfrak{D}_i\right)\right)\sum f(\bar{x})G(I)\right|$$
$$=\left|(\mathfrak{D}')\sum f(\bar{x})G(I)-(\mathfrak{D}'')\sum f(\bar{x})G(I)\right|<\varepsilon .$$

By Cauchy's Criterion, $f \in HS(G,B)$. □

Theorem 3.7. *Let* $E=[\bar{a},\bar{b}]\subset \mathbf{R}^n$ *be a cell,* $B\subseteq E$ *and* $f,G:E\to\mathbf{R}$ *are functions on a cell* E *. If* f *is a continuous function on* E *and* G *is a bounded variation function on a cell* E *then* f *is Henstock-Stieltjes integrable with respect to* G *on a cell* E *.*

PROOF. Since G is a bounded variation function on a cell E, then there exists a constant $M=2V(G;E)+1>0$ such that $|G(I)|\le M$, for every $I\in\mathfrak{I}(E)$, with $V(G;E)$ is variation of G on a cell E. We choose $\{\varepsilon_n\}$ be an arbitrary decreasing sequence of real number and converges to 0. Since f is a continuous function on a closed and bounded cell E then f uniformly continuous on E. So for every natural number n there exists positif constan $\delta_n>0$, such that for every $\bar{x},\bar{y}\in E$, $\|\bar{x}-\bar{y}\|\le\delta_n$, we have

$$|f(\bar{x})-f(\bar{y})|<\frac{\varepsilon_n}{M}.$$

Without loose of the meaning, assume that $\delta_{n+1}<\delta_n$, for every $n\in N$, then we choose arbitrary Perron δ_n -fine partition $\mathfrak{D}_n=\{(I,\bar{x})\}$ on a cell E. Construct $\{S_n\}$ such that

$$S_n=(\mathfrak{D}_n)\sum f(\bar{x})G(I) \text{ for every } n\in N .$$

Furthermore, we choose $m,n\in\mathbf{N}$ with $m\ge n$. Then we have that $\delta_m\le\delta_n$ for every $m\ge n$. From the definition we have that $\delta_n\le\delta_m$, for every $n\ge m$, so we have, every Perron δ_n -fine partition on E also Perron δ_m- fine partition on E, when $n \geq m$. We have for every Perron δ_m-fine partition $\mathfrak{D}_m = \{(I', \bar{x}')\}$ on E also Perron δ_n-fine partition on E. Therefore, if $I \cap I' \neq \phi$ then there exists point $\bar{z} \in I \cap I'$ and we have

$$\begin{aligned} |f(\bar{x}) - f(\bar{x}')| &= |f(\bar{x}) - f(\bar{z}) + f(\bar{z}) - f(\bar{x}')| \\ &\leq |f(\bar{x}) - f(\bar{z})| + |f(\bar{z}) - f(\bar{x}')| \\ &< \frac{\varepsilon_m}{M} + \frac{\varepsilon_n}{M} \leq 2\frac{\varepsilon_n}{M}. \end{aligned}$$

Without loose of the meaning, we choosed $\varepsilon_n = \frac{1}{n}$, for $n = 1,2,3,...$ then $\{\varepsilon_n\}$ is a decreasing sequence of real number and converges to 0. With Archimedian properties, that given any real number $\varepsilon > 0$, there exists natural number $K > \frac{1}{\varepsilon}$, then for every $m, n \geq K$, $n > m$, we have $\varepsilon_m < \varepsilon_n < \varepsilon_K < \varepsilon$. For every $m, n \geq K$, $n > m$, put one positive constant δ_K on E, then we have

$$|S_m - S_n| = |(\mathfrak{D}_m)\sum f(\bar{x}')G(I') - (\mathfrak{D}_n)\sum f(\bar{x})G(I)|$$

$$\leq |(\mathfrak{D}_m)\sum f(\bar{x}')G(I') - (\mathfrak{D}_m)\sum f(\bar{z})G(I') + (\mathfrak{D}_m)\sum f(\bar{z})G(I') - (\mathfrak{D}_n)\sum f(\bar{x})G(I)|$$
$$< |(\mathfrak{D}_m)\sum f(\bar{x}')G(I') - (\mathfrak{D}_m)\sum f(\bar{z})G(I')| + |(\mathfrak{D}_n)\sum f(\bar{z})G(I) - (\mathfrak{D}_n)\sum f(\bar{x})G(I)|$$
$$= |(\mathfrak{D}_m)\sum (f(\bar{x}') - f(\bar{z}))G(I')| + |(\mathfrak{D}_n)\sum (f(\bar{z}) - f(\bar{x}))G(I)|$$
$$\leq V(G;E)\frac{\varepsilon_m}{M} + V(G;E)\frac{\varepsilon_n}{M} \leq 2V(G;E)\frac{\varepsilon_n}{M} < \varepsilon_n < \varepsilon.$$

Thus obtained that $\{S_n\}$ is a Cauchy sequence in $\mathbf{R}$ then $\{S_n\}$ is convergence sequence. So, there exists a real number A as the limit of $\{S_n\}$. That means for every $\varepsilon > 0$ as the above, there exist natural number $n' > \frac{2}{\varepsilon}$, then for every $n \in \mathbf{N}$, $n > n'$ we have $|S_n - A| < \frac{\varepsilon}{2}$. Let $\delta = \delta_{n'}$ is a positive function on E, then for every Perron δ-fine Partition $\mathfrak{D} = \{(I, \bar{x})\}$, on E we have

$$\begin{aligned} |(\mathfrak{D})\sum f(\bar{x})G(I) - A| &= |(\mathfrak{D})\sum f(\bar{x})G(I) - S_{n'} + S_{n'} - A| \\ &\leq |(\mathfrak{D})\sum f(\bar{x})G(I) - S_{n'}| + |S_{n'} - A| \\ &< \frac{1}{n'} + \frac{\varepsilon}{2} \quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

So we can proved that A is the Henstock-Stieltjes integral value of f with respect to G on E. That means $f \in HS(G, E)$. □

From the above theorem we have that the sufficient condition to guarantee the existence of the Henstock-Stieltjes integral function with respect to a bounded variation function G on a cell $E \subset \mathbf{R}^n$ if f a continuous function.

Some convergence theorem of the integral are still prevailed on the Henstock-

Stieltjes integral with respect to the integrator of bounded variation function G on a cell $E \subset \mathbf{R}^n$. In the theory of integral, the convergence theorems give some sufficience conditions for a sequence of Henstock-Stieltjes integrable functions $\{f_k\}$ such that

$$\lim_{k\to\infty}(HS)\int_E f_k dG = (HS)\int_E \lim_{k\to\infty} f_k dG.$$

In this section, we discuss some convergence theorem such as the Uniformly Convergence Theorem and the Monotone Convergence Theorem on the Henstock-Stieltjes integral with respect to the integrator G on a cell $E \subset \mathbf{R}^n$.

Theorem 3.8. (*The Uniformly Convergence Theorem)* *Let $E = [\overline{a}, \overline{b}] \subset \mathbf{R}^n$ be a cell, $G : E \to \mathbf{R}$ is a bounded variation function on a cell E and $f_k : E \to \mathbf{R}$ with $f_k \in HS(G,E)$ for every $k \in N$. If $\{f_k\}$ uniformly converges to a function f on a cell E then $f \in HS(G,E)$. Furthermore,*

$$(HS)\int_E fdG = \lim_{k\to\infty}(HS)\int_E f_k dG.$$

PROOF Since G is a bounded variation function on a cell E, then there exists a constant $M > 0$ such that for every $\mathcal{P} = \{I_i; i = 1,2,3,....,m\}$ partition on a cell E we have $(\mathcal{P})\sum|G(I)| = \sum_{i=1}^{m}|G(I_i)| \le M$. Let $\varepsilon > 0$ be given. Without loose of the meaning, assume that $\{f_k\}$ uniformly converges to a function f on a cell E then there is exists natural number $k_o \in \mathbf{N}$, then for every $\overline{x} \in E$ and $k \ge k_o$, we have $|f_k(\overline{x}) - f(\overline{x})| < \dfrac{\varepsilon}{3.2^{k+1}(M+1)}$.

Since $f_k \in HS(G,E)$ for every $k \in \mathbf{N}$, then there is exists real A_k then for every $\varepsilon > 0$ at the above, there is δ positive function on acell E such as for every Perron δ - fine partitions $\mathfrak{D} = \{(I, \overline{x})\}$ on a cell E we have

$$\left|(\mathfrak{D})\sum f_k(\overline{x})G(I) - A_k\right| < \varepsilon.$$

We defined $\{A_k\}$ is the sequence, where

$$A_k = (HS)\int_E f_k dG$$

for every k.

So $A_k \in \mathbf{R}$ for every k. Furthermore we must show that $\{A_k\}$ is a Couchy sequence. merupakan barisan Cauchy. For every $\varepsilon > 0$ at the above, we choose $k_o = n_o$. Then for every natural number $m, n \ge n_o$ and for arbitrary Perron δ - fine partition $\mathfrak{D}$ on a cell E, we have

$$|A_n - A_m| = \left|(HS)\int_E f_n dG - (HS)\int_E f_m dG\right|$$

$$< \left|(HS)\int_E f_n dG - (\mathfrak{D})\sum f_n(\bar{x})G(I)\right| + \left|(\mathfrak{D})\sum f_n(\bar{x})G(I) - (\mathfrak{D})\sum f(\bar{x})G(I)\right|$$

$$+ \left|(\mathfrak{D})\sum f(\bar{x})G(I) - (\mathfrak{D})\sum f_m(\bar{x})G(I)\right| + \left|(\mathfrak{D})\sum f_m(\bar{x})G(I) - (HS)\int_E f_m dG\right|$$

$$< \varepsilon + \left|(\mathfrak{D})\sum \left(f_n(\bar{x}) - f(\bar{x})\right)G(I)\right| + \left|(\mathfrak{D})\sum \left(f(\bar{x}) - f_m(\bar{x})\right)G(I)\right| + \varepsilon$$

$$< \varepsilon + \sum_{n=1}^{\infty} \frac{\varepsilon}{3.2^n(M+1)}\left|G(I)\right| + \sum_{m=1}^{\infty} \frac{\varepsilon}{3.2^m(M+1)}\left|G(I)\right| + \varepsilon$$

$$< \varepsilon + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \varepsilon < 3\varepsilon .$$

So, for every k, $\{A_k\}$ is a Cauchy sequence in $\mathbf{R}$ and convergence sequence. So, there exists a real number A as the limit of $\{A_k\}$. Katakan konvergen ke suatu $A \in \mathbf{R}$. That means for every $\varepsilon > 0$ as the above, there exist natural number n_1, then for every $k \geq n_1$ we have

$$\left|A_k - A\right| < \varepsilon .$$

We put $k^* = \max\{n_o, n_1\}$ then we defined a positive function $\delta(\bar{x}) = \delta_{k^*}(\bar{x})$ for every $\bar{x} \in E$ and for every Perron δ_{k^*}-fine partition $\mathfrak{D}^*$ on a cell E we have

$$\begin{aligned}\left|(\mathfrak{D})\sum f(\bar{x})G(I) - A\right| &\leq \left|(\mathfrak{D})\sum f(\bar{x})G(I) - (\mathfrak{D})\sum f_k(\bar{x})G(I)\right| \\ &\quad + \left|(\mathfrak{D})\sum f_k(\bar{x})G(I) - A_k\right| + \left|A_k - A\right| \\ &\leq \left|(\mathfrak{D})\sum f((\bar{x}) - f_k(\bar{x}))G(I)\right| + \varepsilon + \varepsilon \\ &< \sum_{k=1}^{\infty} \frac{\varepsilon}{3.2^k(M+1)}\left|G(I)\right| + 2\varepsilon < \frac{\varepsilon}{3} + 2\varepsilon < 3\varepsilon .\end{aligned}$$

That means

$$A = (HS)\int_E f dG .$$

Moreover, we have

$$(HS)\int_E f dG = \lim_{k\to\infty}(HS)\int_E f_k dG \square$$

Next, we will discuss the monotone convergence theorem in the Henstock-Stieltjes integral on $E \subset \mathbf{R}^n$.

Theorem 3.9. (***Monotone Convergence Theorem*****)** *Let* $E = [\bar{a}, \bar{b}] \subset \mathbf{R}^n$ *be a cell,* $G : E \to \mathbf{R}$ *is a bounded variation function on a cell* E*, and* $f_k, f : E \to \mathbf{R}$ *for every* $k \in N$ *. If*

(i). $f_k \to f$ *almost everywhere in* E *as* $k \to \infty$ *with* $f_k \in HS(G,E)$ *for every* $k \in N$ *,*

(ii). $\{f_k\}$ *is monotone in* E , and

(iii). $\{F_k(E)\}$*, the sequence of the value of Henstock-Stieltjes integral* f_k *with respect to integrator of* G *on* E*, converges to* A *as* $k \to \infty$*, then* $f \in HS(G,E)$ *to* A *on* E *and*

$$(HS)\int_E f dG = \lim_{k\to\infty}(HS)\int_E f_k dG .$$

PROOF Since G is a bounded variation function on a cell E, then there exists a constant $M > 0$ such that for every $\mathcal{P} = \{I_i; i = 1,2,3,....,m\}$ partition on a cell E we have

$$(\mathcal{P})\sum |G(I)| = \sum_{i=1}^{m} |G(I_i)| \le M .$$

For $\{f_k\}$ is increasing monotone sequence in a cell E. Let $\varepsilon > 0$ be given. Based on (i) $f_k \in HS(G,E)$ for every $k \in \mathbf{N}$. Then for every $k \in \mathbf{N}$ and $\varepsilon > 0$ at the above, there is positive function δ_k-fine on E such as for every $\mathfrak{D}_k = \{(I,\bar{x})\}$ Perron δ_k - fine partition on E we have

$$(\mathfrak{D}_k)\sum |f_k(\bar{x})G(I) - F_k(I)| < \frac{\varepsilon}{3.2^{k+1}} .$$

Furthermore we have

$$|f_k(\bar{x})G(I) - F_k(I)| \le (\mathfrak{D}_k)\sum |f_k(\bar{x})G(I) - F_k(I)| < \frac{\varepsilon}{3.2^{k+1}}$$

for every $k \in \mathbf{N}$.

Based on (ii), $\{f_k\}$ increasing monotone sequence in a cell E and $f_k \in HS(G,E)$. By Theorem 3.6.. we have $f_k \in HS(G,I)$ where $I \subseteq E$ for every $k \in \mathbf{N}$. Moreover, we have

$$F_k(I) = (HS)\int_I f_k dG \le (HS)\int_I f_{k+1} dG = F_{k+1}(I) \qquad (3.1)$$

Then $\{F_k(I)\}$ is increasing monotone sequence and bounded. That means $\{F_k(I)\}$ converges. Suppose $\{F_k(I)\}$ converges to $F(I)$. Based on (iii) $F_k(E)$ converges to real nuber A. Moreover we have

$$\lim_{k\to\infty} F_k(E) = \lim_{k\to\infty}(HS)\int_E f_k dG = A .$$

So there is positive number k_o such that for every $k \ge k_o$ we have

$$|F_k(E) - A| < \frac{\varepsilon}{3} .$$

Next, without loose of generality we suppose $f_k \to f$ almost everywhere on E for $k \to \infty$. Then for every $\bar{x} \in E$ and $\varepsilon > 0$ at the above, we can choose the positive integer $K(\bar{x},\varepsilon) \ge k_o$, then for every $k \ge K(\bar{x},\varepsilon)$ we have

$$|f_k(\bar{x}) - f(\bar{x})| < \frac{\varepsilon}{3.2^k(M+1)} .$$

Furthermore, we defined a positive function δ on a cell E, with $\delta(\bar{x}) = \delta_{K(\bar{x},\varepsilon)}(\bar{x})$, for every $\bar{x} \in E$. So, for any Perron δ - fine partition

$\mathfrak{D} = \{(I, \bar{x})\}$ on a cell E we have

a) There is infinite associated point in $\mathfrak{D}$, so we can choose $p = \min\{K(\bar{x}, \varepsilon) ; \bar{x} \in (I, \bar{x}) \in \mathfrak{D}\}$. By formula (3.1) we have
$$F_p(I) \leq F_{K(\bar{x},\varepsilon)}(I),$$
for every $I \subseteq E$.
Furthermore, we have
$$F_p(E) = (\mathfrak{D})\sum F_p(I) \leq (\mathfrak{D})\sum F_{K(\bar{x},\varepsilon)}(I) \leq F(E) = A.$$
Then we have
$$A - (\mathfrak{D})\sum F_{K(\bar{x},\varepsilon)}(I) \leq A - (\mathfrak{D})\sum F_p(I).$$

b) For every pairs of a finite point-cell $(I, \bar{x}) \in \mathfrak{D}$ are Perron δ - fine partition on a cell I and union for all the pairs of a finite point-cell $(I, \bar{x}) \in \mathfrak{D}$ are Perron δ - fine partition on a cell E.
Moreover, we have
$$\begin{aligned}
\left|(\mathfrak{D})\sum f(\bar{x})G(I) - A\right| &\leq (\mathfrak{D})\sum \left|f(\bar{x})G(I) - f_{K(\bar{x},\varepsilon)}(\bar{x})G(I)\right| \\
&\quad + (\mathfrak{D})\sum \left|f_{K(\bar{x},\varepsilon)}(\bar{x})G(I) - F_{K(\bar{x},\varepsilon)}(E)\right| + \left|(\mathfrak{D})\sum F_{K(\bar{x},\varepsilon)}(E) - A\right| \\
&\leq (\mathfrak{D})\sum \left|\left(f(\bar{x}) - f_{K(\bar{x},\varepsilon)}(\bar{x})\right)G(I)\right| + (\mathfrak{D})\sum \left|f_{K(\bar{x},\varepsilon)}(\bar{x})G(I) - F_{K(\bar{x},\varepsilon)}(E)\right| \\
&\quad + \left|(\mathfrak{D})\sum F_p(I) - A\right| \\
&< \sum_{K(\bar{x},\varepsilon)=1}^{\infty} \frac{\varepsilon}{3.2^{K(\bar{x},\varepsilon)}(M+1)}\left|G(I)\right| + \sum_{K(\bar{x},\varepsilon)=1}^{\infty} \frac{2\varepsilon}{3.2^{K(\bar{x},\varepsilon)+1}} + \frac{\varepsilon}{3} \\
&< \sum_{K(\bar{x},\varepsilon)=1}^{\infty} \frac{\varepsilon}{3.2^{K(\bar{x},\varepsilon)}(M+1)} M + \sum_{K(\bar{x},\varepsilon)=1}^{\infty} \frac{2\varepsilon}{3.2^{K(\bar{x},\varepsilon)+1}} + \frac{\varepsilon}{3} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

That means $f \in HS(G, E)$ to the number A on a cell E. Then by (iii), we have
$$\lim_{k\to\infty} F_k(E) = \lim_{k\to\infty}(HS)\int_E f_k\,dG = A = F(E) = (HS)\int_E f\,dG.$$
In the other word, we have
$$(HS)\int_E f\,dG = \lim_{k\to\infty}(HS)\int_E f_k\,dG.$$
By remaining this condition: "if $\{f_k\}$ decreasing monotone on a cell E then $\{-f_k\}$ increasing monotone on a cell E", that the Theorema 3.9. also prevailed for $\{f_k\}$ decreasing monotone on a cell E. □

4. Concluding Remarks

The properties of the Henstock-Stieltjest integral in the real line still hold in the $n-$ dimensional Euclidean space, with respect to the integrator of bounded variation function. Some theorems include the Cauchy's Criterion (Theorem 3.4.), the existence of the Henstock-Stieltjes integral function with respect to a bounded variation function G on a cell $E \subset \mathbf{R}^n$ (Theorem 3.7.), the Uniformly

Convergence Theorem (Theorem 3.8.) and the Monotone Convergence Theorem (Theorem 3.9.) are still valid on the Henstock-Stieltjes integral with respect to the integrator of bounded variation function G on a cell $E \subset \mathbf{R}^n$.

References

[1] Indrati, C.R., The Henstock-Kurzweil Integral in n- dimentional Euclidean Space, *Disertation*, Gadjah Mada University, Yogyakarta, 2002.

[2] Lee P.Y., *Lanzhou Lectures on Henstock Integration*, World Scientific, Singapore, 1989.

[3] Lee P.Y. & Výborný, R., *The Integral: An Easy Approach after Kurzweil and Henstock*, Cambridge University Press, 2000.

[4] Lim, J. H., Yoon, S., L., and Eun, G.S, *On Henstock-Stieltjes Integral*, Kangweon – Kyungki Mathematics Journal 6, No:1, pp. 87-96, 1998.

[5] Pfeffer, W.F., *The Riemann Approach to Integration*, Cambridge University Press, New-York, USA, 1993.

ON UNIFORM CONVERGENCE OF SINE INTEGRAL WITH CLASS p-SUPREMUM BOUNDED VARIATION FUNCTIONS

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Abstract. Recently, the monotone decreasing coefficients of sine series has been generalized by classes of general monotone sequences, non-one sided bounded variation sequences, supremum bounded variation sequences. Further class of general monotone functions as counterpart of class of general monotone sequences can be extended to classes of *p*-general mononotone functions and *p*-non-one sided bounded variation functions. In this paper we construct class of *p*-supremum bounded variation functions, a generalization of *p*-general monotone functions and *p*-non-one sided bounded variation functions. The second we study the uniform convergence of sine integral with this class.

Key words and Phrases: *p*-supremum bounded variation functions, sine integral, uniform convergence.

1. Introduction

Chaundy and Jollife [1] proved the following classical theorem:

Theorem 1.1. *Suppose that* $\{a_k\} \subset [0,\infty)$ *is decreasingly tending to zero. A necessary and sufficient conditions for the uniform convergence of the series*

$$\sum_{k=1}^{\infty} a_k \sin kx \tag{1.1}$$

is $\lim_{k\to\infty} k\, a_k = 0.$

Denote by *MS* the monotone decreasing sequences of coefficients (1.1), the class of *MS* has been generalized by many researchers such as Tikhonov [11], Zhou et. al. [12], and Korus [7]. These classes are *GMS* (general monotone sequences), *NBVS* (non-one sided bounded variation sequences), *MVBVS* (mean value bounded variation sequences) and *SBVS* (supremum bounded variation sequences). Zhou et al [12] proved that $MS \subsetneq GMS \subsetneq NBVS \subsetneq MVBVS$ and Korus [7] showed that $MVBVS \subsetneq SBVS$.

Futhermore, Liflyand and Tikhonov [9] generalized *GMS* to $\mathcal{GMS}_p$ (*p-general monotone sequences*), $1 \le p < \infty$. Let $a = \{a_n\}$ and $\beta = \{\beta_n\}$ be two sequences of complex and non-negatif numbers, respectively, a pair $(a, \beta) \in \mathcal{GMS}_p$ if there exists $C > 0$ such that

$$\left(\sum_{k=n}^{2n-1} |a_k - a_{k+1}|^p\right)^{1/p} \le C\beta_n,$$

for p, $1 \le p < \infty$. Theorem 1.2. was proved by Dyachenko and Tikhonov [2] and it is generalization of Theorem 1.1.

Theorem 1.2. *Let* $(a, \beta) \in \mathcal{GMS}_p$, *if for p*=1 *and* $n\beta_n = o(1)$, *then the series* (1.1) *converges uniformly on* $[0, 2\pi]$.

Imron, et al [6] generalized *MVBVS* and *SBVS* to $\mathcal{MVBVS}_p$ (*p-mean value bounded variation sequences*) and $\mathcal{SBVS}_p$ (*supremum bounded variation sequences*) respectively. A *pair* $(a, \beta) \in \mathcal{MVBVS}_p$ if there exist $C > 0$ and $\lambda \ge 2$ such that

$$\left(\sum_{k=n}^{2n-1} |a_k - a_{k+1}|^p\right)^{1/p} \le C \sum_{k=[n/\lambda]}^{[\lambda n]} \beta_k,$$

for p, $1 \le p < \infty$and $(a, \beta) \in \mathcal{SBVS}_p$ if there exist positif constant C and $\gamma \ge 1$ such that

$$\left(\sum_{k=n}^{2n-1} |a_k - a_{k+1}|^p\right)^{1/p} \le \frac{C}{n}\left(\sup_{m \ge [n/\gamma]} \sum_{k=m}^{2m} \beta_k\right),$$

for p, $1 \le p < \infty$. Alittle modification of definition of $\mathcal{SBVS}_p$ gives a class $\mathcal{SBVS}2_p$.A pair (a, β) is said to be *p-supremum bounded variation sequences of second type* , written $(a, \beta) \in \mathcal{SBVS}2_p$, if there exist positive constant C and $\{b(k)\} \subset [0, \infty)$ tending monotonically to infinity depending only on $\{a_k\}$, such that

$$\left(\sum_{k=n}^{2n-1} |a_k - a_{k+1}|^p\right)^{1/p} \le \frac{C}{n}\left(\sup_{m \ge b(n)} \sum_{k=m}^{2m} \beta_k\right),$$

holds for p, $1 \le p < \infty$. Imron, et al [5] have shown that $\mathcal{MVBVS}_p \subsetneq \mathcal{SBVS}_p \subsetneq \mathcal{SBVS}2_p$.

The following theorem was proved by Imron, et. al. [4] and it is more general than Theorem 1.2.

Theorem 1.3. *Let* $(a, \beta) \in \mathcal{SBVS}2_p$, *if* $n^{1-1/p} \sup_{m \ge b(n)} \sum_{k=m}^{2m} \beta_k = o(1)$, *for* $1 \le p < \infty$ **,** *then series* (1.1) *converges uniformly on* $[0, 2\pi]$.

Recently certain efforts have been made in extending the class of monotone sequences to class of monotone functions. Liflyand and Tikhonov [9] introduced the class of *GM (general monotone functions)* as counterpart of *GMS* and generalized *GM* to $\mathcal{GM}_p$, $1 \le p < \infty$. Furthermore Imron et. al. [3] generalized $\mathcal{GM}_p$ to $\mathcal{NBV}_p$ (*p*-non-one sided bounded variation) . The definition of $\mathcal{GM}_p$ and $\mathcal{NBV}_p$ are stated in Definition 1.4. For the rest of the discussion, we assumed *f,* defined on $\mathbb{R}_+$, is locally absolute continuous on $\mathbb{R}_+$ where $\mathbb{R}_+ = (0, \infty)$.

Definition 1.4. *Let* f *and* β *be complex valued and non-negatif functions, respectively and defined* on $\mathbb{R}_+$,

(i) A *pair* (f, β) *is said to be p-general monotone, written* $(f, \beta) \in \mathcal{GM}_p$, *if there exists a positive constant* C *such that*

$$\left(\int_x^{2x} |f'(t)|^p \, dt \right)^{1/p} \leq C\beta(x), \quad for\ all\ x \in \mathbb{R}_+,$$

(ii) A *pair* (f, β) *is said to be p-non-one sided bounded variation, written* $(f, \beta) \in \mathcal{NBV}_p$, *if there exists a positive constant* C *such that*

$$\left(\int_x^{2x} |f'(t)|^p dt \right)^{1/p} \leq C\big(\beta(x) + \beta(2x)\big), \quad for\ all\ \ x \in \mathbb{R}_+,$$

for $1 \leq p < \infty$.

Furthemore Moricz [10] defined $MBVF(\mathbb{R}_+)$, and Korus [8] defined $SBVF(\mathbb{R}_+)$ and $SBVF2(\mathbb{R}_+)$ in Definition 1.5.

Definition 1.5. *A complex valued function f belongs to the class of*

(i) *mean bounded variation functions* $(MBVF(\mathbb{R}_+)$ *if there exist constants* $C > 0$, $A > 0$ *and* $\lambda \geq 2$ *depending only on f such that*

$$\int_a^{2a} |f'(x)| \, dx \leq \frac{C}{a} \int_{\frac{a}{\lambda}}^{\lambda a} |f(x)| \, dx$$

for $a > A$.

(ii) *supremum bounded variation functions* $(SBVF(\mathbb{R}_+)$ *if there exist constants* $C > 0$, $A > 0$ *and* $\lambda \geq 2$ *depending only on f such that*

$$\int_a^{2a} |f'(x)| \, dx \leq \frac{C}{a} \sup_{b \geq a/\lambda} \left(\int_b^{2b} |f(x)| \, dx \right)$$

for $a > A$.

(iii) *supremum bounded variation functions of second type* $(SBVF2(\mathbb{R}_+)$ *if there exist constants* $C > 0$, $A > 0$ *and a function* $B\colon \mathbb{R}_+ \to [0, \infty)$ *tending monotonically to infinity, depending only on f such that*

$$\int_a^{2a} |f'(x)| \, dx \leq \frac{C}{a} \sup_{b \geq B(a)} \left(\int_b^{2b} |f(x)| \, dx \right)$$

for $a > A$.

The counterpart of *SBVS* called $SBVF(\mathbb{R}_+)$ will be used to observe the sufficient condition of the integral (1.2) where $f\colon \mathbb{R}_+ \to \mathbb{C}$ and $t \in \mathbb{R}_+$ to be uniform convergence of the sine integral

$$\int_{0}^{\infty} f(x)\sin tx\,dx\,. \tag{1.2}$$

The uniform convergence of (1.2) means the uniform of

$$\int_{0}^{a} f(x)\sin tx\,dx\,,\text{as } a \to \infty.$$

Due to (1.2), Moricz [10] used a member of $MVBV(\mathbb{R}_+)$ to proved Theorem 1.6.

Theorem 1.6. *Assume $f \in MVBV(\mathbb{R}_+)$,*

(i) if $f\colon \mathbb{R}_+ \to \mathbb{C}$, and

$$|xf(x)| < \infty \text{ as } x \to \infty, \tag{1.3}$$

then the integral (1.2) *is uniformly bounded as $b \to \infty$.*

(ii) If $f\colon \mathbb{R}_+ \to [0,\infty)$ and integral (1.2) *is uniformly bounded, then condition* (1.3) *is satisfied.*

In the present paper, we shall construct classes of *p-mean value bounded variation functions* ($\mathcal{MVBVF}_p$), *p-supremum bounded variation functions* ($\mathcal{SBVF}_p$) and *p-supremum bounded variation functions of second type* ($\mathcal{SBVF}2_p$). Our goal is to extend the Theorem 1.3. from sine series (1.1) to sine integral (1.2) by the results of the new construction and using Theorem 1.6.

2. Main Results

In this section, we construct class of *p-mean value bounded variation functions, p-supremum bounded variation functions* and *p-supremum bounded variation functions of the second type.* Furthemore we investigate the relations among those classes and also study other properties of class of *p-supremum bounded variation functions of the second type*. The Goal of this paper is to study the uniformly corvergence of sine integral (1.2) in Theorem 2.11 and uniformly bounded of integral (1.2) in Theorem 2.12. We always assume that each such complex valued function f is defined on $\mathbb{R}_+$ measurable and locally absolute continuous on $\mathbb{R}_+$.

Definition 2.1. *Let f and β be complex valued and non-negatif functions, respectively and defined* on $\mathbb{R}_+$. *A pair (f,β) is said to be p-mean bounded variation function, written $(f,\beta) \in \mathcal{MVBVF}_p(\mathbb{R}_+)$, if there exist constants $C > 0$, $A > 0$ and $\lambda \geq 2$ depending only on f such that*

$$\left(\int_{a}^{2a} |f'(x)|^p dx\right)^{1/p} \leq \frac{C}{a}\int_{\frac{a}{\lambda}}^{\lambda a} \beta(x)\,dx \; for\; a > A,$$

for $a > A$ and $1 \leq p < \infty$.

Definition 2.2. *Let f and β be complex valued and non-negatif functions,* *respectively and defined* on $\mathbb{R}_+$. A *pair* (f, β) *is said to be p-supremum bounded variation functions, written* $(f, \beta) \in \mathcal{SBVF}_p(\mathbb{R}_+)$ *if there exist constants* $C > 0$, $A > 0$ *and* $\lambda \geq 2$ *depending only on f such that*

$$\left(\int_a^{2a} |f'(x)|^p \, dx\right)^{1/p} \leq \frac{C}{a} \sup_{b \geq \frac{a}{\lambda}} \left(\int_b^{2b} \beta(x) \, dx\right)$$

for $a > A$ *and* $1 \leq p < \infty$.

Futhermore, we define a class is more general than class $\mathcal{SBVF}_p(\mathbb{R}_+)$, that is class of $\mathcal{SBVF}2_p(\mathbb{R}_+)$ (*p-supremum bounded variation functions of second type*), which is stated in Definition 2.3.

Definition 2.3. *Let* f *and* β *be complex valued and non-negatif functions, respectively and defined* on $\mathbb{R}_+$, *for* $1 \leq p$, *a pair* (f, β) *is said to be p-supremum bounded variation functions of second type, written*$(f, \beta) \in \mathcal{SBVF}2_p(\mathbb{R}_+)$, *if there exist constants* $C > 0, A > 0$ *and a function* $B: \mathbb{R}_+ \to [0, \infty)$ *tending monotonically to infinity, depending only on f such that*

$$\left(\int_a^{2a} |f'(x)|^p \, dx\right)^{1/p} \leq \frac{C}{a} \sup_{b \geq B(a)} \left(\int_b^{2b} \beta(x) \, dx\right),$$

for $a > A$ *and* $1 \leq p < \infty$.

The following properties, we prove that classes of $\mathcal{GM}_p(\mathbb{R}_+)$ and $\mathcal{MVBVF}_p(\mathbb{R}_+)$ are subclass of $\mathcal{SBVF}2_p(\mathbb{R}_+)$.

Theorem 2.4. *If* $1 \leq p < \infty$, *then*$\mathcal{MVBVF}_p(\mathbb{R}_+) \subseteq \mathcal{SBVF}_p(\mathbb{R}_+)$.

PROOF. Let $(f, \beta) \in \mathcal{MVBVF}_p(\mathbb{R}_+)$, and μ be the integer such that $2^\mu \leq \lambda^2 < 2^{\mu+1}$. By Definition 2.1 there exist constants $C > 0, A > 0$and Theorem 2.3 [8] by replacing *f* with β, we have

$$\left(\int_a^{2a} |f'(x)|^p dx\right)^{1/p} \leq \frac{C}{a} \int_{\frac{a}{\lambda}}^{\lambda a} \beta(x) \, dx$$

$$\leq \frac{C}{a} \sum_{k=0}^{\mu} \int_{2^k a/\lambda}^{2^{k+1} a/\lambda} \beta(x) dx$$

$$\leq \frac{(\mu + 1)C}{a} \sup_{b \geq a/\lambda} \left(\int_b^{2b} \beta(x) dx\right)$$

Therefore $(f, \beta) \in \mathcal{SBVF}_p(\mathbb{R}_+)$, with constant $(\mu + 1)C$, A and λ.□

Theorem 2.5. *If* $1 \le p < \infty$, *then* $\mathcal{GM}_p(\mathbb{R}_+) \subseteq \mathcal{NVBV}_p(\mathbb{R}_+) \subseteq \mathcal{MVBVF}_p(\mathbb{R}_+)$.

PROOF. From definition 1.1, it is clear that

$$\mathcal{GM}_p(\mathbb{R}_+) \subseteq \mathcal{NVBV}_p(\mathbb{R}_+).$$

Second, we proof that $\mathcal{NVBV}_p(\mathbb{R}_+) \subseteq \mathcal{MVBVF}_p(\mathbb{R}_+)$. By idea proof of Theorem 3 introduced by Moricz [10], if $f \in \mathcal{NVBV}(\mathbb{R}_+)$ obtained

$$\int_a^{2a} |f'(x)|\, dx \le C(|f(a)| + |f(2a)|) \le \frac{4C}{a} \int_{\frac{a}{\lambda}}^{\lambda a} |f(x)|\, dx$$

for $\lambda = 4$ and a positive constant $4C$. Since $(f, \beta) \in \mathcal{NVBV}_p(\mathbb{R}_+)$

$$\left(\int_a^{2a} |f'(x)|^p\, dx \right)^{1/p} \le C(\beta(a) + \beta(2a)) \le \frac{4C}{a} \int_{\frac{a}{\lambda}}^{\lambda a} \beta(x)\, dx$$

for non-negatif function β defined on $\mathbb{R}_+$. Therefore $(f, \beta) \in \mathcal{MVBVF}_p(\mathbb{R}_+)$

Theorem 2.6. *If* $1 \le p < q < \infty$, *then*$\mathcal{SBVF}_q(\mathbb{R}_+) \subseteq \mathcal{SBVF}_p(\mathbb{R}_+)$

PROOF. Let $(f, \beta) \in \mathcal{SBVF}_q(\mathbb{R}_+)$, by Definition 2.2 there exist constants $C > 0, A > 0$ and for $a > A$ such that

$$\left(\int_a^{2a} |f'(x)|^q \right)^{1/q} \le \frac{C}{a} \int_{\frac{a}{\lambda}}^{\lambda a} \beta(x).$$

For $1 \le p < q < \infty$ and applying Holder's inequality to $|f'|^p$ and 1,

$$\int_a^{2a} |f'(x)|^p\, dx = \int_a^{2a} |f'(x)|^p . 1\, dx$$

$$\le \left(\int_a^{2a} |f'(x)|^{pq/p} \right)^{p/q} \left(\int_a^{2a} 1 dx \right)^{1-p/q} = \left(\int_a^{2a} |f'(x)|^q \right)^{p/q} a^{1-p/q}.$$

Therefore

$$\left(\int_a^{2a} |f'(x)|^p \right)^{1/p} \le a^{1/p-1/q} \left(\int_a^{2a} |f'(x)|^q \right)^{1/q} \le \frac{C'}{a} \int_{\frac{a}{\lambda}}^{\lambda a} \beta(x).$$

Thus $(f, \beta) \in \mathcal{SBVF}_p(\mathbb{R}_+)$.□

Theorem 2.7. *If* $1 \le p < \infty$, *then* $\mathcal{SBVF}_p(\mathbb{R}_+) \subseteq \mathcal{SBVF}2_p(\mathbb{R}_+)$.

PROOF. Take $B(a) = a/\lambda$, it is easy to see that $\mathcal{SBVF}_p(\mathbb{R}_+) \subseteq \mathcal{SBVF}2_p(\mathbb{R}_+)$.

Theorem 2.8. *If* $1 \le p < q < \infty$, *then* $\mathcal{SBVF}2_q(\mathbb{R}_+) \subseteq \mathcal{SBVF}2_p(\mathbb{R}_+)$

PROOF. This proof is similar with the proof of Theorem 2.6 by substituting

a/λ with $B(a)$.□

Some properties of$\mathcal{SBVF}2_p(\mathbb{R}_+)$,$1 \le p < \infty$, are stated below

Theorem 2.9. *If* $(f,\beta) \in \mathcal{SBVF}2_p(\mathbb{R}_+)$ and $t^{1-1/p} \sup\limits_{b \ge B(a)} \left(\int_b^{2b} \beta(x)\, dx\right)$*decreasing monotone for* $t \in \mathbb{R}_+$*then*

$$t\int_t^{\infty} |f'(x)|\, dx \le Ct^{\mathbf{1-1/p}} \sup_{b \ge B(t)} \left(\int_b^{2b} \beta(x)\, dx\right)$$

holds for p, $1 \le p < \infty$.

PROOF. Let $(f,\beta) \in \mathcal{SBVF}2_p(\mathbb{R}_+)$ with constants *C, A* and a function *B.* By definition 2.2, we have

$$t\int_t^{\infty} |f'(x)|\, dx \le t\sum_{j=0}^{\infty} \int_{2^j t}^{2^{j+1}t} |f'(x)|\, dx$$

$$\le t\sum_{j=0}^{\infty} \left(\int_{2^j t}^{2^{j+1}t} |f'(x)|^p\, dx\right)^{1/p} \left(2^j t\right)^{\mathbf{1-1/p}}$$

$$\le \sum_{j=0}^{\infty} \left(\int_{2^j t}^{2^{j+1}t} |f'(x)|^p\, dx\right)^{\frac{1}{p}} \frac{\left(2^j t\right)^{2-\frac{1}{p}}}{2^j}$$

$$\le C' \sum_{j=0}^{\infty} \frac{\left(2^j t\right)^{\mathbf{1-1/p}}}{2^j} \sup_{b \ge B(2^j t)} \left(\int_b^{2b} \beta(x)\, dx\right)$$

$$\le C'\, t^{\mathbf{1-1/p}} \sup_{b \ge B(t)} \left(\int_b^{2b} \beta(x)\, dx\right) \sum_{j=0}^{\infty} \frac{1}{2^j}$$

$$\le Ct^{\mathbf{1-1/p}} \sup_{b \ge B(t)} \left(\int_b^{2b} \beta(x)\, dx\right)$$

The proof is complete. □

Theorem 2.10. *If* $(f,\beta) \in \mathcal{SBVF}2_p(\mathbb{R}_+)$ *and* $t^{1-1/p} \sup\limits_{b \ge B(t)} \left(\int_b^{2b} \beta(x)\, dx\right)$ *tending to* 0 *for* $t \to \infty$, *then*

$$\int_t^{\infty} |f'(x)|\, dx < \infty$$

for p, $1 \le p < \infty$.

PROOF. *Let* $t^{1-1/p} \sup\limits_{b \ge B(a)} \left(\int_b^{2b} \beta(x)\, dx\right)$ tending to 0 and we denote

$$\rho_t = \sup_{s \ge t}\left(s^{1-1/p} \sup_{b \ge B(t)}\left(\int\limits_b^{2b} \beta(x)\, dx\right)\right).$$

then $\rho_t \to 0$. Let $(a, \beta) \in \mathcal{SBVS}2_p(\mathbb{R}_+)$ and from proof of Theorem 2.9. we obtained

$$t\int\limits_t^\infty |f'(x)|\, dx \le C' \sum_{j=0}^{\infty} \frac{\left(2^j t\right)^{1-1/p}}{2^j} \sup_{b \ge B(2^j t)}\left(\int\limits_b^{2b} \beta(x)\, dx\right)$$

$$\le C' \sum_{j=0}^{\infty} \frac{d_t}{2^j} \le C d_t, C = 2C'$$

Therefore

$$\int\limits_t^\infty |f'(x)|\, dx \le C\frac{d_t}{t} < \infty.$$

The proof is complete. □

In the properties bellow, we always assume that for $I: \mathbb{R}_+ \to \mathbb{R}_+$,

$$I.f \text{ locally integrable on } \mathbb{R}_+ \tag{2.1}$$

Theorem 2.11. *Let* $(f, \beta) \in \mathcal{SBVF}2_p(\mathbb{R}_+)$ *and* $t^{1-1/p} \sup\limits_{b \ge B(t)} \left(\int_b^{2b} \beta(x)\, dx\right)$ *is tending to 0 for* $t \to \infty$, $1 \le p < \infty$,

(i) *If condition*

$$xf(x) \to 0 \text{ as } x \to \infty, \tag{2.2}$$

is satisfied, then sine integral of (1.2) *converges uniformly in t.*

(ii) *If* $f: \mathbb{R}_+ \to [0, \infty)$ *and* sine *integral of* (1.2) *converges uniformly in t, then condition* (2.2) *is satisfied.*

PROOF.

(i) We denote

$$\rho_t = \sup_{s \ge t}\left(s^{1-1/p} \sup_{b \ge B(s)}\left(\int\limits_b^{2b} \beta(x)\, dx\right)\right).$$

Since $t^{1-1/p} \sup\limits_{b \ge B(t)} \left(\int_b^{2b} \beta(x)\, dx\right) \to 0$, then $\rho_t \to 0$. Let $(f, \beta) \in \mathcal{SBVF}2_p(\mathbb{R}_+)$, we may use proof of Theorem 2.10. for every $\varepsilon > 0$ there exists $t_0 = t_0(\varepsilon) \in \mathbb{R}_+$, such that

$$t\int_t^\infty |f'(x)|\,dx < \varepsilon \tag{2.3}$$

for all $t \geq t_0$. Furthemore, first for $a < b < 1/t$ and by condition (2.1) and (2.2) we have

$$\left|\int_a^b f(x)sintx\,dx\right| \leq t\int_a^b x|f(x)|\,dx \leq \frac{1}{1/t}\int_0^{1/t} x|f(x)|\,dx < \varepsilon$$

Second, for he case $\frac{1}{t} < a < b$, integration by parts and (2.3) we have

$$\left|\int_a^b f(x)sintx\,dx\right| \leq \left|\left[-f(x)\frac{cos\,tx}{t}\right]_a^b + \int_a^b f'(x)\frac{cos\,tx}{t}\right|$$

$$\leq \frac{1}{t}\left\{|f(a)| + |f(b)| + \int_a^b |f'(x)|\,dx\right\}$$

$$\leq a|f(a)| + b|f(b)| + a\int_a^\infty |f'(x)|\,dx < 3\varepsilon$$

From first and second case, the integral (1.2) converges uniformly in *t.*

(ii) Let integral (1.2) converges uniformly in *t*, this means that for every $\varepsilon > 0$, there exists $a_0 = a_0(\varepsilon) \in \mathbb{R}_+$ such that

$$\left|\int_a^b f(x)\sin tx\,dx\right| < \varepsilon, \qquad \text{for all } b > a > a_0 \text{ and } t.$$

We start with the equality

$$f(y) - f(a) = \int_a^{2a} f'(x)\,dx, \qquad 0 < a \leq y \leq 2a. \tag{2.4}$$

Since $(f,\beta) \in \mathcal{SBVSF}2_p(\mathbb{R}_+)$ and from (2.4) that

$$f(a) \leq f(y) + \int_a^{2a} |f'(x)|\,dx \leq f(y) + \left(\int_a^{2a} |f'(x)|^p\,dx\right)^{1/p}\left(a^{1-1/p}\right) \tag{2.5}$$

$$\leq f(y) + Ca^{-1/p}\sup_{b\geq B(a)}\left(\int_b^{2b} \beta(x)\,dx\right)$$

We integrate (2.5) with respect to y over the interval $[a, 2a]$ to obtain

$$af(a) \leq \int_a^{2a} f(y)\,dy + Ca^{1-1/p}\sup_{b\geq B(a)}\left(\int_b^{2b} \beta(x)\,dx\right) \tag{2.6}$$

Since$t^{1-1/p} \sup_{b\geq B(t)} \left(\int_b^{2b} \beta(x)\,dx\right)$ tending to 0 for $t \to \infty$, from (2.6) we have

$$af(a) \leq \int_a^{2a} f(y)\,dy. \tag{2.7}$$

Set

$$t(a) = \frac{\pi}{4a}, \quad a \in \mathbb{R}_+,$$

then

$$\frac{\pi}{4} \leq t(a)x \leq \frac{\pi}{2}, \qquad if \;\; a \leq x \leq 2a.$$

From these inequality and (2.7)we have

$$\left|\int_a^{2a} f(x)\sin tx\,dx\right| \geq \sin\frac{\pi}{4}\int_a^{2a} f(x)\,dx \geq af(a)\sin\frac{\pi}{4},$$

for all $a \in \mathbb{R}_+$.

Since integral (1.2) is supposed to converge uniformly in *t,* therefore

$$xf(x) \to 0 \;\text{ as }\; x \to \infty.$$

The proof is complete. □

Theorem 2.12. *Let* $(f,\beta) \in \mathcal{SBVF}2_p(\mathbb{R}_+)$ and $t^{1-1/p} \sup_{b\geq B(t)} \left(\int_b^{2b} \beta(x)\,dx\right)$ *is tending to 0 for* $t \to \infty$, $1 \leq p < \infty$,

(i) *If condition*

$$|xf(x)| < \infty \;\textit{as}\; x \to \infty, \tag{2.8}$$

then the sine integral of (1.2) *is uniformly bounded as* $b \to \infty$.

(ii) *If* $f: \mathbb{R}_+ \to [0,\infty)$ *and the sine integral of* (1.2) *is uniformly bounded, then condition* (2.8) *is satisfied.*

PROOF.

(i) We may use proof of Theorem 2.10, there exists $t_0 = t_0(\varepsilon) \in \mathbb{R}_+$, such that

$$t\int_t^{\infty} |f'(x)|\,dx < Cd_t \tag{2.9}$$

for all $t \geq t_0$. Furthemore, first for $a < b < 1/t$ and by condition (2.1) and (2.8) there exists constant $B > 0$ such that

$$\left|\int_a^b f(x) sintx\, dx\right| \le t\int_a^b x|f(x)|\, dx \le \frac{1}{\frac{1}{t}}\int_0^{\frac{1}{t}} x|f(x)|\, dx \le B.$$

Second, for case $\frac{1}{t} < a < b$, integration by parts and (2.9) we have

$$\left|\int_a^b f(x) sintx\, dx\right| \le \left|\left[-f(x)\frac{cos\ tx}{t}\right]_a^b + \int_a^b f'(x)\frac{cos\ tx}{t}\right|$$

$$\le \frac{1}{t}\left\{|f(a)| + |f(b)| + \int_a^b |f'(x)|\, dx\right\}$$

$$\le a|f(a)| + b|f(b)| + a\int_a^\infty |f'(x)|\, dx < 2B + Cd_t$$

From first and second case, the partial integrals of (1.2) uniformly are bounded.

(ii) Let partial integrals of (1.2) are buniformly bounded, this means that for every constant $B > 0$, there exists $b_0 = b_0(B) \in \mathbb{R}_+$ such that

$$\left|\int_a^b f(x)\sin tx\, dx\right| \le B, \qquad \text{for all } b > b_0 \text{ and } t. \tag{2.10}$$

Similarly with proof of Theorem 2.11. (*ii*) wehave

$$\left|\int_a^{2a} f(x)\sin tx\, dx\right| \ge \sin\frac{\pi}{4}\int_a^{2a} f(x)\, dx \ge af(a)\sin\frac{\pi}{4},$$

for all $a \in \mathbb{R}_+$.

By inequality (2.9) we have

$$xf(x) \le B \ \text{ as } \ x \to \infty.$$

The proof is complete. □

Corollary 2.13. *Let* $(f,\beta) \in \mathcal{SBVF}_p(\mathbb{R}_+)$ and $t^{1-1/p}\sup\limits_{b\ge a/\lambda}\left(\int_b^{2b}\beta(x)\, dx\right)$ *is tending to 0 for* $t \to \infty,\ 1 \le p < \infty$.

(i) *If condition* (2.8) *satisfies, then the integral of* (1.2) is*uniformly bounded as* $b \to \infty$.

(ii) *If* $f: \mathbb{R}_+ \to [0, \infty)$ *andthe sine integral of (1.2) is uniformly bounded, then condition* (2.8) *is satisfied.*

Corollary 2.14. *Let* $(f, \beta) \in \mathcal{SBVF}_p(\mathbb{R}_+), 1 \leq p < \infty$ and $t^{1-1/p}\left(\int_{a/\lambda}^{\lambda a} \beta(x)\, dx\right)$ *is tending to 0 for* $t \to \infty$.

(i) *If condition* (2.8) *satisfies, then the sine integral of* (1.2) *is uniformly bounded as* $b \to \infty$.

(ii) *If* $f: \mathbb{R}_+ \to [0, \infty)$ *and the sine integral of* (1.2) *is uniformly bounded, then condition* (2.8) *is satisfied.*

3. Concluding Remarks

In this paper we have introduced the class $\mathcal{SBVF}2_p$. We have investigated that

(i) The class of $\mathcal{SBVF}2_P(\mathbb{R}_+)$ is more general than class $\mathcal{GM}_p(\mathbb{R}_+)$ by Theorem 2.4, Theorem 2.5 and Theorem 2.7.

(ii) The sufficient condition of Uniformly convergence of (1.2) in Theorem 2.11 are$(f, \beta) \in \mathcal{SBVF}2_p(\mathbb{R}_+)$for $1 \leq p < \infty$, $t^{1-1/p} \sup\limits_{b \geq B(t)} \left(\int_b^{2b} \beta(x)\, dx\right)$ is tending to *0* for $t \to \infty$ and $xf(x) \to 0$ as $x \to \infty$.

(iii) The sufficient condition of Uniformly bounded of (1.2) in Theorem 2.12 are $(f, \beta) \in \mathcal{SBVF}2_p(\mathbb{R}_+)$ for $1 \leq p < \infty$, $t^{1-1/p} \sup\limits_{b \geq B(t)} \left(\int_b^{2b} \beta(x)\, dx\right)$ is tending to *0* for $t \to \infty$ and $|xf(x)| < \infty$ as $x \to \infty$.

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References

[1] Chaundy, T. W. and Jollife, A.E., The Uniform Convergence of certain class trigonometric series, *Proc. London, Soc*. 15, 214-116, 1916.

[2] Dyachenko, M. and Tikhonov, S., General monotone sequences and convergence of trigonometric series, in: *Topics in Classical Analysis and Applications in Honor of Daniel Waterman ,World Scientific, Hakensack, NJ,* pp. 88-101, 2008.

[3] Imron, M. A., Indrati, C, R. and Widodo, On *p*-non One Sided Bounded Variation Sequences and Functions, *Proceeding of 2nd Basic Science International Conference,* M46-M50,FMIPA, UB, Malang, 2012.

[4] Imron, M.A., Indrati, C. R. and Widodo, On uniform convergence of trigonometric series under *p*-supremum bounded variation condition, *Proceeding of the third Basic Science International Conference,* M041-M043,FMIPA, UB, Malang, 2013.

[5] Imron, M. A., Indrati, C. R. and Widodo, Some properties of class of p-Supremum bounded Variation sequences, *Int.Journal of Math. Analysis,* Vol 7, no.35, 1703-1713, 2013. online:http//dx.doi.org/10.12988/ijma.2013.3494.

[6] Imron, M.A., Indrati, C. R. and Widodo, Relasi Inklusi pada Klas Barisan *p*-Supremum Bounded variation, *Jurnal Natural A ,*No 1, Vol 1, (1-6) FMIPA, UB, Malang, 2013.

[7] Korus, P., Remark On the uniform And L^1-Convergence of Trigonometric Series, *Acta Math. Hungar*, 128(4), 2010.

[8] Korus, P., On the uniform Convergence Of special sine integrals, *Acta Math. Hungar*, 133 (I), 82-91, 2011.

[9] Liflyand, E. and Tikhonov, S., A concept of general monotonicity an application, *Math. Nachr*, 284, No.8-9, 1083-1098, 2011.

[10] Moricz, F., On the uniform convergence of sine integral, *J.Math Anal Appl,* 254, 213-219, 2009.

[11] Tikhonov, S., Best approximation and moduli of Smoothness computation and Equivalence Theorems*, Journal of Approximation Theory*, 153, 19-39, 2008.

[12] Zhou, S.P., Zhou, P. and Yu, D. S., Ultimate generalization to monotonicity for Uniform Convergence of Trigonometric Series, online:http//arxiv.org/abs/math/0611805v1

APPLICATION OF OPTIMAL CONTROL OF THE CO_2 CYCLED MODEL IN THE ATMOSPHERE BASED ON THE PRESERVATION OF FOREST AREA

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Abstract. The CO_2 content in the atmosphere cycles related to the plant assimilation process, respiration process and many other human activities. There are two scenarios to be observed in this paper. The first scenario is consider the nature into atmosphere and biosphere zone, while the second one is consider the nature into the atmosphere, biosphere and sea-zone. The growth of the CO_2 content in those zones is controlled by propose the forest area preservation (u_1) and the successess rate of the reforestation program (u_2). The ***optimal control*** of it is also designed to satisfy the ***stability*** of temperature and pressure interaction in the atmosphere. The ***Hamiltonian function*** is generated in order to find the values of u_1 and u_2 using the ***Lagrange equations*** that must satisfy the ***stationer condition*** and both ***state and co-state equations***. It could be seen from the research result that not only the CO_2 content in the atmosphere and biosphere zones must be controlled but also the CO_2 content in the sea zone. This effort will reduces the CO_2 content in each zones and keeps the stable temperature and pressure interaction in the atmosphere.
Key words and Phrases: Hamiltonian function, Lagrange equations, optimal control, state and co-state equations, stationer codition.

1. Introduction

The growth of CO_2 emition rate is one of the most strategical issues topic in climate changing research. Forest and green zone area, as a main biotic resources, are able to change CO_2 to be O_2 such that could play a role to maintain a stable interaction of temperature and pressure as main climate unsures. The climate changing actually indicates the forest unability to reduce the CO_2 emition rate such that we come to the global warming phenomena.

Haneda [2] stated that Global warming describes the dynamic of temperature that gives impact to the climate changing. The impacts could be seen from the changing of wind and climate pattern, atmospheric hurricane and hydrological cycle [11]. Tjasyono stated on [14] that *green house effect*, i.e CO_2, gives a significant contribution to the growth of CO_2 content and pressure in the atmosphere. Soedomo [11] supports [14] by stated that it already indicated the growth of temperature because the CO_2 content rise continuesly. Global warming could be avoided by arrange the forest area to control the CO_2 content and to resist the growth of temperature in the asmosphere. The problem is how to control it based on the preservation of forest area. Mathematically this problem is stated as the CO_2 content controlling problem. The related optimal control could be got by building such *performance index* that satisfy *Pontryagin* maximum principe in [6], [7] and [12].

It could be understand that the contribution of CO_2 content in the atmosphere will determine the temperature profile such that causes the dynamics of the other main climate unsure, i.e. pressure. The dynamics of temperature and pressure that caused by the growth of CO_2 content in the atmosphere is very important to be observed. Refering to [3], the interaction between both main climate unsures is derived from the thermodynamics law and is analysed by consider the represented mathematical model that describe the interaction between temperature and pressure in the atmosphere. The most important result of [2] is the dependencies of the stability of the system with respect to the parameter that represent the caloric exchange rate to the atmosphere. This parameter must be bounded by the difference between the the heat capacity and the specific volume of gas. This is the main reason to consider the contribution of CO_2 content in the atmosphere to the system by extend the system in a dynamical system form of temperature, pressure and CO_2 content in the atmosphere.

The observation of the extended system in [10] shows that the caloric exchange rate also appear in the requirement of its stability with the same criteria of stability sistem in [3]. It means that the system keeps the parameter to be the determined factor that bring the system into an stable or unstable condition. Another condition must be investigated by consider the eigen value of the linearization of the nonlinear system represented the extended system. The behaviour of CO_2 content in the atmosphere is evaluated to decide wether the system must be revised or not.

In case that the system is needed to be revised, we must consider the cycle of CO_2 content in the universe that [1] devided it into atmosphere, biosphere and sea zone. The cycle describes the CO_2 transport that delivered from the sea, as the place of respiration and assimilation of plant in the earth, to the atmosphere and driven to the biosphere. If the interpretation of the solution of the system of [4] is not represent the reduction of CO_2 content, this paper propose to revised the third equation of [4] by the derived equations that build in [1]. The new tentative system is observed until the phenomena of the CO_2 reduction in the atmosphere could be found. This paper also revise the compartements that built in [1] by promote a related parameter due to the CO_2 reduction program in biosphere.

The CO_2 reduction program in biosphere, that being considered in this paper, is the green zone preservation and the reforestation program. A cycled CO_2 content model is proposed by [8]. To reduce the growth CO_2 content the model must be controlled by proposing the forest area preservation (u_1) and the succesess rate of the reforestation program (u_2). This program will revise the changing of CO_2 content in the biosphere with respect to the time changing. Mathematically it will cause the appearance of new term on the related equation.

2. Main Results

2.1 The Initial Model

A dynamical system that represents the interaction of temperature, pressure and CO_2 content in the atmosphere is stated in [4] by

$$\dot{T} = \frac{Q[\propto pQ + C_p - TQ]}{C_p[C_p - K \propto - Q]}$$

$$\dot{p} = \frac{Q[K - pQ]}{[C_p - K \propto - Q]}$$

$$\dot{x} = b\ x\left(1 - c\left(\frac{T}{p}\right)\right) - \frac{ex}{1+x} \quad .... (1)$$

Where T : temperature, p : pressure, x : CO_2 content in the atmosphere, C_p: heat capacity in fixed pressure condition, α: specific volume of gas, Q: caloric exchange rate and b, c and e: related parameters with respect to the production and consumption of CO_2.

An ***optimal control*** of the initial model is designed by generating the ***Hamiltonian*** function to find the values of control parameters u_1 and u_2 using the ***Lagrange*** equations that must satisfy the ***stationer condition*** and both ***state and co-state*** equations. The parameters respectively represents the growth of the CO_2 content in the biosphere zone and the rate of forest area preservation identified by the successess rate of the reforestation program. The optimal solution of the optimal controlled problem is arranged such that the ***stability*** between temperature and pressure interaction in the atmosphere could be reached. The system stated on (1) is controlled by places parameter A in the third equation of system (1) that interpreted as the atmospheric CO_2 absorption program that is designed to minimized the CO_2 production in the atmosphere with performance index $J(A) = \int_0^{t_f} f(x, A, t)dt = \frac{1}{2}\int_0^{t_f}(c_1A^2 + c_2x^2)dt$. A ***Hamiltonian*** function $H = f(X, A, t) + \sum_{i=1}^{3}\lambda_i g_i(X, A, t)$, , $X = (T, p, x)^T$ $a_1 \le A \le b_1$, $g_1(X, A, t) = \frac{Q[\propto pQ + C_p - TQ]}{C_p[C_p - K\propto - Q]}$, $g_2(X, A, t) = \frac{Q[K - pQ]}{[C_p - K\propto - Q]}$, $g_3(X, A, t) = b\,A\,x\left(1 - c\left(\frac{T}{p}\right)\right) - \frac{ex}{1+x}$ is generated to find the value of A using the ***Lagrange*** equations $L = f(X, A, t) + \sum_{i=1}^{3}\lambda_i g_i(X, A, t) + \omega_{11}(t)(b_1 - A) + \omega_{12}(t)(A - a_1)$, $\omega_{11}(t), \omega_{12}(t) \ge 0$, $\omega_{11}(t)(b_1 - A) = 0$, $\omega_{12}(t)(A - a_1) = 0$. If the ***stationer condition*** $\frac{\partial L}{\partial A} = 0$ satisfy both respectively ***state and co-state*** equations $\dot{X} = \left(g_1(X, A, t), g_2(X, A, t), g_3(X, A, t)\right)^T$ and $-\dot{\lambda}^T = \frac{\partial L}{\partial X}$ we got :

$$A = \begin{cases} \frac{-\lambda_3 b\,x + \lambda_3 b\,x\,c\left(\frac{T}{p}\right)}{c_1}, \text{jika } a_1 \le \frac{-\lambda_3 b\,x + \lambda_3 bx\,c\left(\frac{T}{p}\right)}{c_1} \le b_1 \\ a_1, \text{ jika } \frac{-\lambda_3 b\,x + \lambda_3 b\,x\,c\left(\frac{T}{p}\right)}{c_1} \ge a_1 \\ b_1, \text{jika } \frac{-\lambda_3 b\,x + \lambda_3 b\,x\,c\left(\frac{T}{p}\right)}{c_1} \le b_1 \end{cases}$$

The simulation of the initial model, with Q = 35, C_p = 36,775, $\alpha = 0{,}017$, K = 0,23, b =1, e =1,8 is refer to [8], [9] and [10] , gives the plots of temperature, pressure and atmospheric CO_2 content shown in Figure 1.

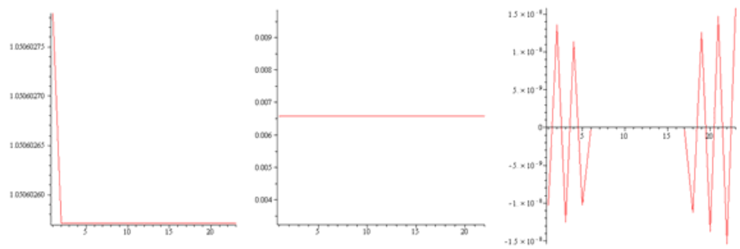

Figure 1. The plots of temperature, pressure and atmospheric CO_2 content

Figure 1 shows that the CO_2 content in the atmosphere already could be reduced but the atmospheric CO_2 is rised and reduced fluctuatively. It means that the stability interaction between the temperature and pressure could be disturbed. This result makes must be revised by redisign the following tentative model.

2.2 The Tentative Model

The tentative model is designed by consider the cycle of CO_2 content in the nature. The CO_2 cycle scheme could be seen in Figure 2.

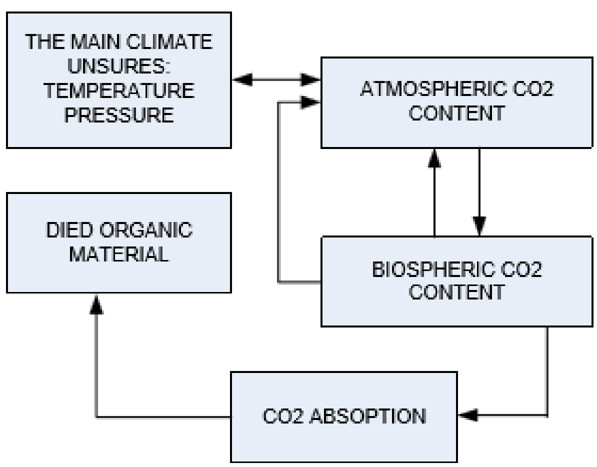

Figure 2. The cycle of CO_2

A revised dynamical system that represents the interaction of temperature, pressure, CO_2 content in the atmosphere, biosphere and the died organic material rate is stated in the following system :

$$\dot{T} = \frac{Q[\propto p\dot{q} + C_p - TQ]}{C_p[C_p - K \propto -Q]}$$

$$\dot{p} = \frac{Q[K - pQ]}{[C_p - K \propto -Q]}$$

$$\dot{X_l} = k_1 X_l X_a - k_1' X_l - k_2 X_l - k_4 \alpha_D X_l$$

$$\dot{X_d} = k_2 X_l - k_3 X_d$$

$$\dot{X_a} = -k_1 X_l X_a + k_1' X_l \text{ .... (2)}$$

Where X_l , X_a, X_d is respectively the CO_2 content in the biosphere, atmosphere and the died organic material rate, Q : caloric exchange rate, C_p : heat capacity, α : gas specific volume, K is positive constant, k_1: respiration intensity, k_1':assimilation intensity, k_2 died organic material rate, k_3 : died organic caloric exchange rate, k_4 : forest area preservation rate and α_D : the successes of reforestation program. The last three equations of system (2) is governed from the cycle of CO_2 in figure 2.

The system is controlled by places two parameters u_1 and u_1 to the system (2) that interpreted as the atmospheric CO_2 absorption programs that is designed to minimized the CO_2 content in the atmosphere and biosphere, with performance index $J(u_1, u_2) = \int_0^{t_f} f(X_l, X_a, u_1, u_2, t)dt = \int_0^{t_f}(c_1 u_1^2 + c_2 u_2^2 + c_3 X_a^2 + c_4 X_l^2)dt$.

A ***Hamiltonian*** function $H = f(X_l, X_a, u_1, u_2, t) + \sum_{i=1}^{5} \lambda_i g_i(X_l, X_a, u_1, u_2, t)$ and $L = f(X_l, X_a, u_1, u_2, t) + \sum_{i=1}^{5} \lambda_i g_i(X, A, t) + \omega_{11}(t)(b_1 - u_1) + \omega_{12}(t)(u_1 - a_1) + \omega_{21}(b_2 - u_2) + \omega_{22}(t)(u_2 - a_2)$, $X = (T, p, X_l, X_d, X_a)^T$ with $g_1(X, u_1, u_2, t) = \frac{Q[\propto p\dot{q} + C_p - TQ]}{C_p[C_p - K\propto -Q]}$, $g_2(X, u_1, u_2, t) = \frac{Q[K - pQ]}{[C_p - K\propto -Q]}$, $g_3(X, u_1, u_2, t) = k_1 X_l X_a - k_1' X_l - k_2 X_l - u_2 k_4 \alpha_D X_l$, $g_4(X, u_1, u_2, t) = u_1 k_2 X_l - k_3 X_d$, $g_5(X, u_1, u_2, t) = -k_1 X_l X_a + u_1 k_1' X_l$, $a_1 \le u_1 \le b_1$, $a_2 \le u_2 \le b_2$, $\omega_{11}(t), \omega_{12}(t), \omega_{21}(t), \omega_{22}(t) \ge 0$, $\omega_{11}(t)(b_1 - u_1) = 0, \omega_{12}(t)(u_1 - a_1) = 0$, $\omega_{21}(t)(b_2 - u_2) = 0, \omega_{22}(t)(u_2 - a_2) = 0$. If the ***stationer condition*** $\frac{\partial L}{\partial u} = 0, u = (u_1, u_2)^T$ satisfy both respectively ***state and co-state*** equations $\dot{X} = \left(g_1(X, u_1, u_2, t), g_2(X, u_1, u_2, t), g_3(X, u_1, u_2, t), g_4(X, u_1, u_2, t), g_5(X, u_1, u_2, t)\right)^T$ and $-\dot{\lambda}^T = \frac{\partial L}{\partial X}$ we got :

$$u_1 = \begin{cases} \frac{-\lambda_4 k_2 X_l - \lambda_5 k_1' X_l}{c_1}, \text{jika } a_1 \le \frac{-\lambda_4 k_2 X_l - \lambda_5 k_1' X_l}{c_1} \le b_1 \\ a_1, \text{ jika } \frac{-\lambda_4 k_2 X_l - \lambda_5 k_1' X_l}{c_1} \ge a_1 \\ b_1, \text{jika } \frac{-\lambda_4 k_2 X_l - \lambda_5 k_1' X_l}{c_1} \le b_1 \end{cases}, \quad u_2 = \begin{cases} \frac{-\lambda_3 k_6 \alpha_D X_l}{c_2}, \text{jika } a_2 \le \frac{-\lambda_3 k_6 \alpha_D X_l}{c_2} \le b_2 \\ a_2, \text{ jika } \frac{-\lambda_3 k_6 \alpha_D X_l}{c_2} \ge a_2 \\ b_2, \text{jika } \frac{-\lambda_3 k_6 \alpha_D X_l}{c_2} \le b_2 \end{cases}$$

The simulation of the initial model, with Q = 35, C_p = 36,775, $\alpha = 0{,}017$, $\beta = 0{,}017, K = 0{,}23$, $k_1' = 0{,}03 k_1 = 4{,}3.10^{-5}, k_2 = 2{,}0.10^{-2}, k_3 = 1{,}6.10^{-2}, k_6 \alpha_D = 0{,}21$ are refer to [8], [9],10] and [1], gives the plots of respectively temperature, pressure, atmospheric and biospheric CO_2 content and the died organic material rate shown in Figure 3.

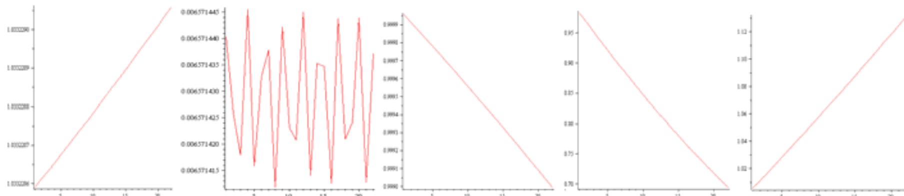

Figure 3. The plots of temperature, pressure, atmospheric CO_2 content, biospheric CO_2 content and the died organic material rate

Figure 3 shows that the CO_2 content in the atmosphere and biosphere already decreases with respect to time while the died organic material rate is still increase that makes the increasing of temperature with a fluctuative pressure condition. It means that a stable interaction of temperature ang pressure is not obtained yet. This result makes must be revised by redesign an extended model that consider the CO_2 content in the sea zone in the nature CO_2 cycle.

2.3 The Extended Model

An extended model is designed by consider the role of CO_2 content in the sea zone in the nature. The extended model is designed by consider the contribution of CO_2 content in the sea zone to the CO_2 cycle in universe. A revised scheme of CO_2 cycle is proposed in Figure 4.

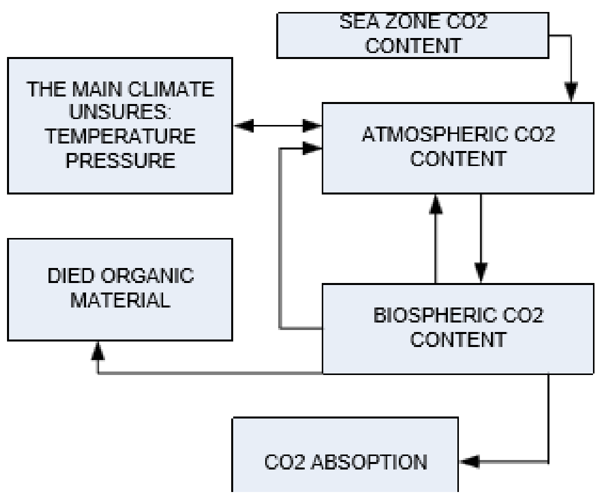

Figure 4. The revised cycle of CO_2

An extended dynamical system that represents the interaction of temperature, pressure, CO_2 content in the atmosphere, biosphere, sea zone and the died organic material rate is stated in the following system :

$$\dot{T} = \frac{Q[\propto p\dot{q} + C_p - TQ]}{C_p[C_p - K \propto -Q]}$$

$$\dot{p} = \frac{Q[K - pQ]}{[C_p - K \propto -Q]}$$
$$\dot{X}_l = k_1 X_l X_a - k_1' X_l - k_2 X_l - k_6 \alpha_D X_l$$
$$\dot{X}_d = k_2 X_l - k_3 X_d$$
$$\dot{X}_s = k_4 (X_a - k_5 X_s)$$
$$\dot{X}_a = -k_1 X_l X_a + k_1' X_l + k_4(-X_a + k_5 X_s) \text{ …….. (3)}$$

where X_s is the CO_2 content in the seazone, k_4 : the sea zone respiration intensity and k_5 : the sea zone assimilation intensity. The last four equations of system (3) is governed from the cycle of CO_2 in figure 4.

The system is controlled by places two parameters u_1 and u_1 to the system (3) that interpreted as the atmospheric CO_2 absorption programs that is designed to minimized the CO_2 content in the atmosphere and biosphere, with performance index $J(u_1,u_2) = \int_0^{t_f} f(X_l, X_a, u_1, u_2, t)dt = \int_0^{t_f} \left(c_1 u_1^2 + c_2 u_2^2 + c_3 X_a^2 + c_4 X_l^2\right)dt$.

A ***Hamiltonian*** function $H = f(X_l, X_a, u_1, u_2, t) + \sum_{i=1}^{6} \lambda_i g_i(X_l, X_a, u_1, u_2, t)$ and $L = f(X_l, X_a, u_1, u_2, t) + \sum_{i=1}^{6} \lambda_i g_i(X, u_1, u_2, t) + \omega_{11}(t)(b_1 - u_1) + \omega_{12}(t)(u_1 - a_1) + \omega_{21}(b_2 - u_2) + \omega_{22}(t)(u_2 - a_2)$, $X = (T, p, X_l, X_d, X_s, X_a)^T$ with $g_1(X, u_1, u_2, t) = \frac{Q[\propto p\dot{q} + C_p - TQ]}{C_p[C_p - K\propto -Q]}$, $g_2(X, u_1, u_2, t) = \frac{Q[K - pQ]}{[C_p - K\propto -Q]}$, $g_3(X, u_1, u_2, t) = k_1 X_l X_a - k_1' X_l - k_2 X_l - u_2 k \ \ \alpha_D X_l$, $g_4(X, u_1, u_2, t) = u_1 k_2 X_l - k_3 X_d$, $g_5(X, u_1, u_2, t) = k_4(u_1 X_a - k_5 X_s)$, $g_6(X, u_1, u_2, t) = -k_1 X_l X_a + u_1 k_1' X_l + k_4(-X_a - k_5 X_s)$, $a_1 \le u_1 \le b_1$, $a_2 \le u_2 \le b_2, \omega_{11}(t), \omega_{12}(t), \omega_{21}(t), \omega_{22}(t) \ge 0$, $\omega_{11}(t)(b_1 - u_1) = 0, \omega_{12}(t)(u_1 - a_1) = 0$, $\omega_{21}(t)(b_2 - u_2) = 0, \omega_{22}(t)(u_2 - a_2) = 0$. If the ***stationer condition*** $\frac{\partial L}{\partial u} = 0, u = (u_1, u_2)^T$ satisfy both respectively ***state and co-state*** equations $\dot{X} = \left(g_1(X,u_1,u_2,t), g_2(X,u_1,u_2,t), g_3(X,u_1,u_2,t), g_4(X,u_1,u_2,t), g_5(X,u_1,u_2,t), g_6(X,u_1,u_2,t)\right)^T$ and $-\dot{\lambda}^T = \frac{\partial L}{\partial X}$ we got :

$$u_1 \begin{cases} \frac{-\lambda_4 k_2 X_l - \lambda_5 k_4 X_a - \lambda_6 k_1' X_l}{c_1}, \text{jika } a_1 \le \frac{-\lambda_4 k_2 X_l - \lambda_5 k_4 X_a - \lambda_6 k_1' X_l}{c_1} \le b_1 \\ a_1, \text{ jika } \frac{-\lambda_4 k_2 X_l - \lambda_5 k_4 X_a - \lambda_6 k_1' X_l}{c_1} \ge a_1 \\ b_1, \text{jika } \frac{-\lambda_4 k_2 X_l - \lambda_5 k_4 X_a - \lambda_6 k_1' X_l}{c_1} \le b_1 \end{cases}, u_2 =$$

$$\begin{cases} \frac{\lambda_3 k_6 \alpha_D X_l}{c_2}, \text{jika } a_2 \le \frac{\lambda_3 k_6 \alpha_D X_l}{c_2} \le b_2 \\ a_2, \text{ jika } \frac{\lambda_3 k_6 \alpha_D X_l}{c_2} \ge a_2 \\ b_2, \text{jika } \frac{\lambda_3 k_6 \alpha_D X_l}{c_2} \le b_2 \end{cases}$$

The simulation of the initial model, with Q = 35, C_p = 36,775, $\alpha = 0{,}017$, β = 0,017, K=0,23, $k_1' = 0{,}03k_1 = 4{,}3.10^{-5}, k_2 = 2{,}0.10^{-2}, k_3 = 1{,}6.10^{-2}, k_6\alpha_D = 0{,}21$ are refer to [8], [9],10] and [1], gives the plots of respectively temperature, pressure, atmospheric, biospheric and sea zone CO_2 content and the died organic material rate shown in Figure 5.

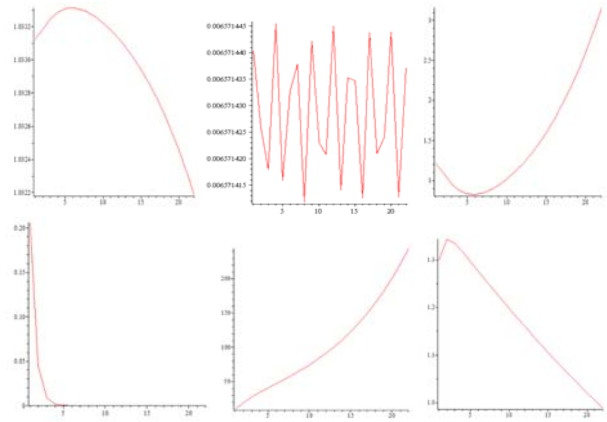

Figure 5. The plots of temperature, pressure, atmospheric, biospheric and sea zone CO_2 content and the died organic material rate of the extended model

Figure 5 shows that the temperature and biospheric CO_2 content will deacreases with respect to time while the decreasing of atmospheric CO_2 content just occur in the first 7 years and increases later on. The increasing of died organic material rate just happen in the begining, after that it will decrease with respect to time. The most important result is the sea zone CO_2 content is still uncontrolled yet. In the next session it will be consider a parameter control in the fifth equation of system (3).

2.4 The Fixed Model

To control the growth of CO_2 content in the atmosphere and sea zone shown in preview section, the system (3) is controlled to minimized the CO_2 content in the atmosphere, biosphere and sea zone, with performance index $J(u_1,u_2) = \int_0^{t_f} f(X_l,X_s,X_a,u_1,u_2,t)dt = \int_0^{t_f}(c_1u_1^2 + c_2u_2^2 + c_3X_a^2 + c_4X_l^2 + c_5X_s^2)dt$. A ***Hamiltonian*** function $H = f(X_l,X_s,X_a,u_1,u_2,t) + \sum_{i=1}^{6}\lambda_i g_i(X,u_1,u_2,t)$ and $L = f(X_l,X_s,X_a,u_1,u_2,t) + \sum_{i=1}^{6}\lambda_i g_i(X,u_1,u_2,t) + \omega_{11}(t)(b_1-u_1) + \omega_{12}(t)(u_1-a_1) + \omega_{21}(b_2-u_2) + \omega_{22}(t)(u_2-a_2), X = (T,p,X_l,X_d,X_s,X_a)^T$, with $g_1(X,u_1,u_2,t) = \frac{Q[\propto p\dot{q}+C_p-TQ]}{C_p[C_p-K\propto-Q]}$, $g_2(X,u_1,u_2,t) = \frac{Q[K-pQ]}{[C_p-K\propto-Q]}$, $g_3(X,u_1,u_2,t) = k_1X_lX_a - k_1'X_l - k_2X_l - u_2k_4\alpha_DX_l$, $g_4(X,u_1,u_2,t) = u_1k_2X_l - k_3X_d$, $g_5(X,u_1,u_2,t) = k_4(u_1X_a - u_1k_5X_s)$, $g_6(X,u_1,u_2,t) = -k_1X_lX_a + u_1k_1'X_l + k_4(-X_a - k_5X_s)$, $a_1 \le u_1 \le b_1$, $a_2 \le u_2 \le b_2$, $\omega_{11}(t),\omega_{12}(t),\omega_{21}(t),\omega_{22}(t) \ge 0$, $\omega_{11}(t)(b_1-u_1) = 0, \omega_{12}(t)(u_1-a_1) = 0$, $\omega_{21}(t)(b_2-u_2) = 0, \omega_{22}(t)(u_2-a_2) = 0$. If the ***stationer condition*** $\frac{\partial L}{\partial u} = 0, u = (u_1,u_2)^T$ satisfy both respectively ***state and co-state*** equations $\dot{X} = \left(g_1(X,u_1,u_2,t), g_2(X,u_1,u_2,t), g_3(X,u_1,u_2,t), g_4(X,u_1,u_2,t), g_5(X,u_1,u_2,t), g_6(X,u_1,u_2,t)\right)^T$ and $-\dot{\lambda}^T = \frac{\partial L}{\partial X}$ we got :

$$u_1 = \begin{cases} \frac{\lambda_3k_2X_l-\lambda_4k_2X_l-\lambda_5k_4X_a+\lambda_3k_1'X_l}{c_1}, \text{jika } a_1 \le \frac{\lambda_3k_2X_l-\lambda_4k_2X_l-\lambda_5k_4X_a+\lambda_3k_1'X_l}{c_1} \le b_1 \\ a_1, \text{ jika } \frac{\lambda_3k_2X_l-\lambda_4k_2X_l-\lambda_5k_4X_a+\lambda_3k_1'X_l}{c_1} \ge a_1 \\ b_1, \text{jika } \frac{\lambda_3k_2X_l-\lambda_4k_2X_l-\lambda_5k_4X_a+\lambda_3k_1'X_l}{c_1} \le b_1 \end{cases},$$

$$u_2 = \begin{cases} \dfrac{\lambda_3 k_6\, \alpha_D X_l}{c_2}, \text{jika } a_2 \leq \dfrac{\lambda_3 k_6\, \alpha_D X_l}{c_2} \leq b_2 \\ a_2, \quad \text{jika } \dfrac{\lambda_3 k_6\, \alpha_D X_l}{c_2} \geq a_2 \\ b_2, \text{jika } \dfrac{\lambda_3 k_6\, \alpha_D\, X_l}{c_2} \leq b_2 \end{cases}$$

The simulation of the initial model, with Q = 35, C_p = 36,775, $\alpha = 0{,}017$, $\beta = 0{,}017$, K=0,23, $k_1' = 0{,}03k_1 = 4{,}3.10^{-5}, k_2 = 2{,}0.10^{-2}, k_3 = 1{,}6.10^{-2}, k_6\alpha_D = 0{,}21$ are refer to [8], [9],10] and [1], gives the plots of respectively temperature, pressure, atmospheric, biospheric and sea zone CO_2 content and the died organic material rate shown in Figure 6.

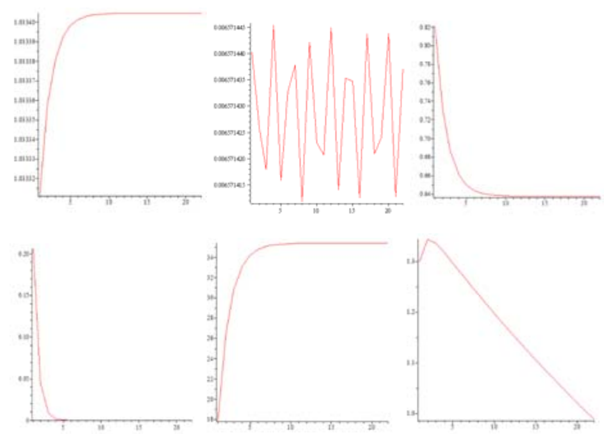

Figure 6. The plots of temperature, pressure, atmospheric CO_2 content, biospheric CO_2 content, sea zone CO_2 content and the died organic material rate

Figure 6 shows that the growth of temperature could be controled properly with with small volatility. It could be concluded from the linear relation between temperature and pressure and seen from the plot of pressure. The CO_2 content in the atmosphere and biospheric also already well controlled while the same behaviour of died organic material rate of the extended system is still found. The most important result is the sea zone CO_2 content already could be controlled.

3. Concluding Remarks

The optimal CO_2 content in the universe that makes the stable interaction between the two main climate unsure, temperature and pressure, is reached when the controller is not only placed in the atmosphere and biosphere zone terms but also in the atmosphere zone term. It could be interpretated that, to have a stable interaction between the temperature and pressure in the atmosphere, it is not enough to control the atmospheric CO_2 content and the biospheric CO_2 content. We can't eliminate the importance role of sea zone, as a place of assimilation and respiration zone of marine biotic, because the area of the sea zone is wider than the continent zone.

Acknowledgement

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References

[1] Eriksson, E. and Welander, P., 1955, *On a Mathematical Model of The Carbon Cycle in The Nature*, International Meteorological Institute, Stockholm.

[2] Haneda, accessed 11 Nopember 2008, *Hubungan Efek Rumah Kaca, Pemanasan Global dan Perubahan Iklim*, http://climatechange.menlh.g.id.

[3] Jaya, A. I., 2010, *Peran Penting Laju Perubahan Kalor pada Model Dinamik Unsur-Unsur Utama Iklim*, Jurnal Ilmiah Matematika dan Terapan Volume 7 no 2 ISSN 1829-8133, UniversitasTadulako, Palu.

[4] Jaya, A.I., 2011, *Mencari Perluasan Model Dinamik Unsur-Unsur Utama Iklim*, Jurnal Ilmiah Matematika dan Terapan Volume 8 no 1 ISSN 1829-8133, Universitas Tadulako, Palu.

[5] Kato, S., Tri W. H dan Joko W., 1998, *Dinamika Atmosfer*, ITB Publisher, Bandung.

[6] Naidu, D.S., 2002, *Optimal Control Systems*, CRS Presses LLC, USA.

[7] Pontryagin, L.S.,et al, 1962, *The Mathematical Theory of Optimal Process*, Vol.4, Interscience.

[8] Ratianingsih, Rina, Jaya, A.I. and Tarende, T.E, 2013, The Design of CO_2 Absorption Model with Respect to The Stability of Temperature and Pressure Interaction in The Atmosphere, Procedia Environmental Sciences CRISU-CUPT VIII International Conference 2013, Kendari.

[9] Ratianingsih, Rina, 2011, *Dynamic Model of Pressure, Temperature and* CO_2 *Concentration : Identification of Stability Parameters Potency*, Presented on The SEAMS-GMU Conference on Mathematics and Its Applications, Universitas Gadjah Mada, Yogyakarta.

[10] Ratianingsih, Rina, 2011, *Identifikasi Model Konsumsi Gas* CO_2 *di Atmosfer untuk Mendapatkan Interaksi Unsur – Unsur Utama Iklim yang Stabil*, akan terbit, Prosiding Seminar Nasional Sains IV, Institut Pertanian Bogor.

[11] Soedomo, M., 2001, *Pencemaran udara*, Kumpulan Karya Ilmiah, Penerbit ITB, Bandung.

[12] Subchan, S and Zbikowski, R, 2009, Computational Optimal Control : Tools and Practices, John Willey & Sons Ltd. Publishing,UK.

[13] Sutimin, 2010, *Model Matematika Konsentrasi Gas Oksigen Terlarut pada Ekosistem Perairan Danau*, UniversitasDiponegoro, Semarang.

[14] Tjasyono, B., 2004, *Klimatologi*, ITB Publisher, Bandung.

COMPARISON OF SENSITIVITY ANALYSIS ON LINEAR OPTIMIZATION USING OPTIMAL PARTITION AND OPTIMAL BASIS (IN THE SIMPLEX METHOD) AT SOME CASES

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Abstract. Sensitivity analysis describes the effects of coefficient changes of a linear optimization problem to the optimal solution. Usually we use the optimal basis approach as in the simplex method. This paper discussed the sensitivity analysis with another approaches: analysis using an optimal partition based on the interior point method to determine the range and shadow price. We then compare the results obtained with those produced by the simplex method with the help of software LINDO 6.1. The results of sensitivity analysis, obtained through the optimal partition approach is more accurate than using the optimal basis approach (the simplex method), especially for cases where the primal or the dual optimal solution is not unique. But when the primal and the dual have a unique optimal solution, the simplex method and the optimal partition approach produce same information.

Key words and Phrases : sensitivity analysis, shadow price, range, optimal partition, optimal basis.

1. Introduction

Linear Optimization (LO) is concerned with the minimization or maximization of a linear function, subject to constraints described by linear equations and/or linear inequalities.

Sensitivity analysis describes the effect of changing the parameters of the linear optimization model, i.e. studying the effect of changing the coefficients of objective function and right-hand side value constraints to the optimal solution. Sensitivity analysis that is used in the classical approach (the simplex method) based on the optimal basis. This paper will present briefly sensitivity analysis by using another approach, the analysis using the unique partition (optimal partition) based on the interior point method. This method is presented by Roos, Terlaky and Vial [1]. By using the optimal partition approach, we determine shadow price and range. For the same problem we also performed a sensitivity analysis using the simplex method with the help of software LINDO 6.1. Then we compare the obtained results.

The structure of this paper is as follows. In section 2, we review shortly the primal-dual problem, optimal partition and optimal sets, range and shadow price, and sensitivity analysis with classical approach. In section 3, we present three cases of LO problems to be analyzed and compared by using optimal partition and by using LINDO 6.1. At the end we give concluding remarks.

2. Sensitivity Analysis

2.1. Primal - Dual

Every linear optimization problem can be modeled mathematically into a form called the primal form (P) and the dual form (D).

The standard form of a primal and a dual form are as follows:
(P) min $\{c^T x : Ax = b, x \geq 0\}$,
(D) max $\{b^T y : A^T y + s = c, s \geq 0\}$,
where $c, x, s \in \mathbb{R}^n$, $b, y \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$ is matrix with rank m.

Suppose the optimal value of (P) and (D) symbolized by $v(b)$ and $v(c)$:
$v(b) = \min \{c^T x : Ax = b, x \geq 0\}$,
$v(c) = \max \{b^T y : A^T y + s = c, s \geq 0\}$.
The feasible regions of (P) and (D) are denoted by P and D, respectively:
$P := \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$,
$D := \{(y, s) \in \mathbb{R}^m : A^T y + s = c, s \geq 0\}$.

If (P) and (D) are feasible then both problems have optimal solutions, and we denote it by P^* and D^*,
$P^* := \{x \in P : c^T x = v(b)\}$
$D^* := \{(y, s) \in D : b^T y = v(c)\}$.

2.2. Optimal Partition and optimal sets

The followings are the theorems that used as base of forming an optimal partition.

Theorem 1. *(Duality Theorem, cf.* [1] *Theorem II.2) If (P) and (D) are feasible then both problems have optimal solutions. Then, if $x \in P$ and $(y, s) \in D$, these are optimal solutions if and only if $x^T s = 0$. Otherwise neither of the two problems has optimal solutions, either both (P) and (D) are infeasible or one of the two problems is infeasible and the other one is unbounded.*

Theorem 2. *(Goldman-Tucker, cf.* [1] *Theorem II.3) If (P) and (D) are feasible then there exists a strictly complementary pair of optimal solutions, that is an optimal solution pair (x^*, s^*) satisfying $x^* + s^* > 0$.*

The optimal partition of (P) and (D) are the partition that splits the index of x (and s) into B and N, as follows:

$$B := \{i : x_i > 0 \text{ for some } x \in P^*\},$$
$$N := \{i : s_i > 0 \text{ for some } (y, s) \in D^*\}.$$

We may check that the duality theorem implies $B \cap N = \emptyset$, and Goldman-

Tucker theorem implies $B \cup N = \{1, 2, ..., n\}$.

We use x_B and x_N as notations refer to the restriction of the vector $x \in \mathbb{R}^n$ to the index set B and N respectively. Similarly, A_B and A_N represent the restriction of A to the columns with indices of set B and N respectively. We then have the following lemma.

Lemma 1.*(cf. [1]) P* and D* can be expressed by the terms of the optimal partition into*

$$P^* = \{x : Ax = b, x_B \geq 0, x_N = 0\},$$
$$D^* = \{(y, s) : A^T y + s = c, s_B = 0, s_N \geq 0\}.$$

2.3. Range and Shadow Price

Sensitivity analysis determines the shadow price and range of all the coefficients b (the value of the right side of primal constraints) and c (the value of the right side dual constraints). In one case, the value of coefficient b or c may be a break point. If the coefficient is a break point, then we have two shadow prices: the left shadow price and right shadow price. If the coefficient is not a break point, then there is a shadow price at an open linearity interval and range of the coefficient is in the linearity interval. Figure 1 shows an example of change in the optimal value for the change in the value of cj (c_j=1 and c_j=2 are break points).

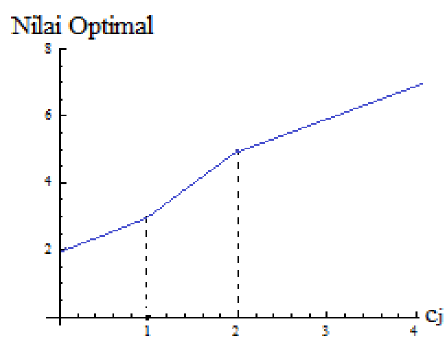

Figure 1. Optimal value function for c_j

Suppose that (P) and (D) are feasible. According to optimal partition approach [1], range of b_i is obtained by minimizing and maximizing b_i over the set

$\{b_i : Ax = b, x_B \geq 0, x_N = 0\}$. (1.1)

Left and right shadow price of b_i are determined by minimizing and maximizing y_i over the set

$\{y_i : A^T y + s = c, s_B = 0, s_N \geq 0\}$. (1.2)

Range of c_j is obtained by minimizing and maximizing the value of c_j over the set

$\{c_j: A^T y + s = c, s_B = 0, s_N \geq 0\}$. (1.3)

Left and right shadow price of c_j are determined by minimizing and maximizing x_j over the set

$\{x_j: Ax = b, x_B \geq 0, x_N = 0\}$. (1.4)

2.4. Sensitivity Analysis with the Classical Approach

Sensitivity analysis with the classical approach based on the simplex method to solve linear optimization problems. The optimal solution of this classical approach is determined by an optimal basis.

Assume that A is a matrix of size mxn and rank (A) = m. Indices of a basis variable is denoted by B'. Then sub-matrix $A_{B'}$ is a non-singular matrix of size mxm with $A_{B'}x_{B'}=b$, $x_{N'}=0$ where N' is the set of non-basis variable index of A. A primal basic solution x can be determined by

$$x = \begin{pmatrix} x_{B'} \\ x_{N'} \end{pmatrix} = \begin{pmatrix} A_{B'}^{-1}b \\ 0 \end{pmatrix}, \ (1.5)$$

and a dual basic solution can be determined by

$$y = A_{B'}^{-T}c_{B'}, \ s = \begin{pmatrix} s_{B'} \\ s_{N'} \end{pmatrix} = \begin{pmatrix} 0 \\ c_{N'}-A_{N'}^{T}y \end{pmatrix}. \ (1.6)$$

Sensitivity analysis with the classical approach uses also formulas (1.5) - (1.8) to determine the range and shadow price, but with the optimal basis partition (B ', N ') instead of (B, N). In fact, P and D may have more than one optimal basis, and therefore this classical approach may also provides different shadow price and range [2].

3. Cases

We consider three cases as follows:

1. Optimal solution of the primal problem is unique and optimal solution of the dual problem is not unique.
2. Optimal solution of the primal problem is not unique and optimal solution of the dual problem is unique.
3. Optimal solution of the primal and the dual problems are unique.

3.1. Case I

Suppose the primal problem (P) is defined as follows:

Min$4x_1 - 5x_2 + 11x_3$

s.t $-x_2 + 3x_3 = 0$

$x_1 - x_2 - x_3 = 1$

$x_1, x_2, x_3 \geq 0$

The dual problem (D) is

Maxy_2

s.t$y_2 \leq 4$

$-y_1 - y_2 \leq -5$

$3y_1 - y_2 \leq 11$

The feasible region of the dual problem is depicted in Figure 2. From Figure 2, it can be seen that the set of optimal solutions of (D) is $D^* = \{(y_1, y_2): 1 \leq y_1 \leq 5, y_2 = 4\}$ and the optimal value is 4 . Slack variable of each of the dual constraints are as follows:

$y_2 + s_1 = 4 \Leftrightarrow s_1 = 4 - y_2$
$-y_1 - y_2 + s_2 = -5 \Leftrightarrow s_2 = -5 + y_1 + y_2$
$3y_1 - y_2 + s_3 = 11 \quad \Leftrightarrow s_3 = 11 - 3y_1 + y_2$

It can be concluded that all the slack can be positive at an optimal solution unless the slack value of the constraint $y_2 \leq 4$, i.e. $s_1 = 0$. This means that the optimal partition of set N is $N=\{2, 3\}$. Hence $B=\{1\}$.
By using Lemma 1, we get:
$P^* = \{x \in P: x_2 = x_3 = 0\}$ and (P) has a unique solution $x = (1, 0, 0)$.

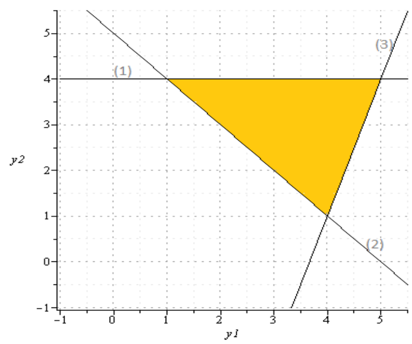

Figure 2. Feasible region of case I.

Next we show examples of finding range and shadow price of $b_1 = 0$ and $c_1 = 4$. The other range and shadow price can be found in the same way.

Range and Shadow Price for $b_1 = 0$
By using (1.1), range b_1 can be determined by minimizing and maximizing b_1 over the set $\{b_1: Ax = b, x_B \geq 0, x_N = 0\}$.
We have $Ax = b$ as follows

$$\begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} b1 \\ 1 \end{bmatrix}.$$

From the above system we get

$0 = b_1,$
$x_1 = 1.$

Hence the range of b_1 is the interval [0, 0]. Therefore $b_1 = 0$ is a break point.
By using (1.2), the shadow price of b_1 can be determined by minimizing and maximizing y_1 over the set $\{y_1 : A^T y + s = c, s_B = 0, s_N \geq 0\}$.
Using that $y \in D^*$, the minimum value of y_1 is 1 and the maximum value is 5, so the shadow price for b_1 is [1, 5].

Range and Shadow Price for $c_1 = 4$
Range of c_1 determine by minimizing and maximizing c_1 over the set $\{c_1: A^T y + s = c, s_B = 0, s_N \geq 0\}$, as in (1.3).
Matrix multiplication of $A^T y + s = c$:

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} y1 \\ y2 \end{bmatrix} + \begin{bmatrix} s1 \\ s2 \\ s3 \end{bmatrix} = \begin{bmatrix} c1 \\ -5 \\ 11 \end{bmatrix}$$

Based on Figure 1, if we eliminate the first constraint, y_2 will be in the interval $[1, \infty)$. By substituting $s_1 = 0$ and y_2 to the first constraint, we get $y_2 = c_1$. This means that $1 \leq c_1 \leq \infty$, hence the range for c_1 is the interval $[1, \infty)$.
By using (1.4) shadow price of c_1 is determine by minimizing and maximizing x_1 over the set $\{x_1: Ax = b, x_B \geq 0, x_N = 0\}$. Because of $x_1 = 1$, then the shadow price of c_1 is 1.

In Table 1, we present range and shadow price of case I which are obtained by using optimal partition approach. We also present range and shadow price obtained from calculation by using LINDO (Table 2). We may see sensitivity analysis of the simplex method (LINDO) did not detect that $b_1 = 0$ is a break point.

Table 1. Range and shadow price obtained from optimal partition approach (Case I)

Coefficient	Range	Shadow price
$b_1 = 0$	0	[1, 5]
$b_2 = 1$	$[0, \infty)$	4
$c_1 = 4$	$[1, \infty)$	1
$c_2 = -5$	$[-9, \infty)$	0
$c_3 = 11$	$[-1, \infty)$	0

Table 2. Range and shadow price obtained from LINDO (Case I)

Coefficient	*Range*	*Shadow price*
$b_1 = 0$	$(-\infty, 0]$	1
$b_2 = 1$	$[0, \infty)$	4
$c_1 = 4$	$[1, \infty)$	1
$c_2 = -5$	$[-9, \infty)$	0
$c_3 = 11$	$[-1, \infty)$	0

3.2. Case II

Suppose the primal problem (P) is defined as follows:
Min $4x_1 + 31x_2 - 5x_3 + 11x_4$
s.t $3x_2 - x_3 + 3x_4 = 0$
$x_1 + 7x_2 - x_3 - x_4 = 1$
$x_1, x_2, x_3, x_4 \geq 0$.

The dual problem (D) is
Maxy_2
s.t$y_2 \leq 4$
$3y_1 + 7y_2 \leq 31$
$-y_1 - y_2 \leq -5$
$3y_1 - y_2 \leq 11$

The feasible region of the dual problem is shown in Figure 3. From Figure 3, we obtain that the optimal solution of (D) is $D^* = \{(y_1, y_2): y_1 = 1, y_2 = 4\}$ and the optimal value is 4. Slack variable of each of the dual constraints are as follows:

$y_2 + s_1 = 4 \;\Leftrightarrow s_1 = 4 - y_2$
$3y_1 + 7y_2 + s_2 = 31 \;\Leftrightarrow s_2 = 31 - 3y_1 - 7y_2$
$-y_1 - y_2 + s_3 = -5 \;\Leftrightarrow s_3 = -5 + y_1 + y_2$
$3y_1 - y_2 + s_4 = 11 \;\Leftrightarrow s_4 = 11 - 3y_1 + y_2$

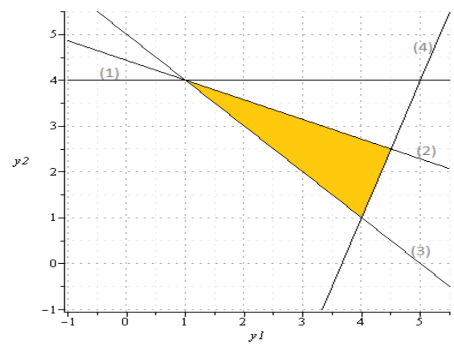

Figure 3. Feasible region of case II.

By substituting $y_1 = 1$, $y_2 = 4$, we obtain the values of each slack. Slack in these constraints: $y_2 \le 4$, $3y_1 + 7y_2 \le 31$ and $-y_1 - y_2 \le -5$ are 0, i.e. $s_1 = s_2 = s_3 = 0$. Hence in the primal problem only x_1, x_2 and x_3 can be positive. Therefore the optimal partition (B, N) is obtained, i.e. N = {4} and B = {1, 2, 3}.

By using Lemma 1, we get:
$P^* = \{x \in P: x_4 = 0\}$ and (P) has not unique solution : $\{ (x_1, x_2, x_3) : (a , \frac{1}{4} - \frac{1}{4} a, 3 (\frac{1}{4} - \frac{1}{4} a)) \}$, $0 \le a \le 1$.
By using the same calculation as in case I, we get ranges and shadow prices of case II (Table 3). Table 4 shows ranges and shadow prices of case II obtained by using LINDO.

Table 3. Range and shadow price obtained from optimal partition approach (Case II).

Coefficient	*Range*	*Shadow price*
$b_1 = 0$	$(-\infty, 3/7]$	1
$b_2 = 1$	$[0, \infty)$	4
$c_1 = 4$	4	[1, 0]
$c_2 = 31$	31	[¼, 0]
$c_3 = -5$	-5	[¾, 0]
$c_4 = 11$	$[-1, \infty)$	0

Table 4. Range and shadow price obtained from LINDO (Case II).

Coefficient	*Range*	*Shadow price*
$b_1 = 0$	$(-\infty, 0]$	1
$b_2 = 1$	$[0, \infty)$	4
$c_1 = 4$	$[1, 4]$	1
$c_2 = 31$	$[31, \infty)$	0
$c_3 = -5$	$[-5, \infty)$	0
$c_4 = 11$	$[-1, \infty)$	0

From Table 3 and Table 4, there are differences in range and shadow price obtained by using optimal partition and the simplex method. At the coefficient $b_1 = 0$, for the same shadow price, the optimal partition detect a greater range. Next, at the coefficients $c_1 = 4$, $c_2 = 31$, and $c_3 = -5$, analysis using the simplex method does not detect any break points.

3.3. Case III

Suppose the primal problem (P) is defined as follows:

$$\begin{aligned}
&\text{Min} 31x_1 - 5x_2 + 11x_3 \\
&\text{s.t } 3x_1 - x_2 + 3x_3 = 0 \\
&\quad 7x_1 - x_2 - x_3 = 1 \\
&\quad x_1, x_2, x_3 \geq 0
\end{aligned}$$

Dual problem (D) is

$$\begin{aligned}
&\text{Max } y_2 \\
&\text{s.t} 3y_1 + 7y_2 \leq 31 \\
&\quad -y_1 - y_2 \leq -5 \\
&\quad 3y_1 - y_2 \leq 11
\end{aligned}$$

The feasible region of the dual problem is shown in Figure 4. From Figure 4, it can be determined that the optimal solution of (D) is $D^* = \{(y_1, y_2): y_1 = 1, y_2 = 4\}$ and the optimal value is 4 . Slack variable of each of the dual constraints are as follows:

$$\begin{aligned}
&3y_2 + 7y_2 + s_1 = 31 \Leftrightarrow s_1 = 31 - 3y_1 - 7y_2 \\
&-y_1 - y_2 + s_2 = -5 \Leftrightarrow s_2 = -5 + y_1 + y_2 \\
&3y_1 - y_2 + s_3 = 11 \quad \Leftrightarrow s_3 = 11 - 3y_1 + y_2
\end{aligned}$$

We can check that at $y_1 = 1$ and $y_2 = 4$, all the slack can be positive except slack in the constraint $3y_1 + 7y_2 \leq 31$ and $-y_1 - y_2 \leq -5$, at the constraints mentioned we have $s_1 = s_2 = 0$. Hence the optimal partition (B, N) is $N = \{3\}$ and $B = \{1, 2\}$.
By using Lemma 1, we obtain:
$P^* = \{x \in P: x_3 = 0\}$ and (P) has a unique solution $x = (¼, ¾, 0)$.

By using the same calculation as before, we get ranges and shadow prices of case III (Table 5). Table 6 shows ranges and shadow prices of case III obtained by using LINDO.

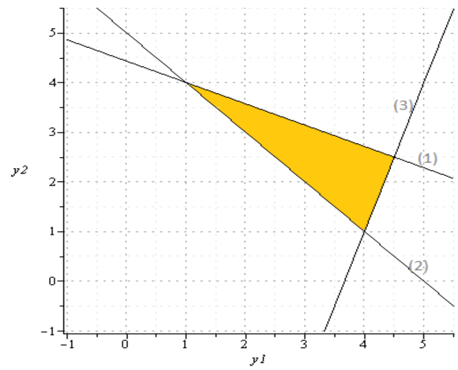

Figure 4. Feasible region of case II

Table 5. Range and shadow price obtained from optimal partition approach (Case III).

Coefficient	*Range*	*Shadow price*
$b_1 = 0$	$(-\infty, 3/7]$	1
$b_2 = 1$	$[0, \infty)$	4
$c_1 = 31$	$[19, \infty)$	¼
$c_2 = -5$	$[-7, \infty)$	¾
$c_3 = 11$	$[-1, \infty)$	0

Table 6. Range and shadow price obtained from LINDO (Case III).

Coefficient	*Range*	*Shadow price*
$b_1 = 0$	$(-\infty, 3/7]$	1
$b_2 = 1$	$[0, \infty)$	4
$c_1 = 31$	$[19, \infty)$	¼
$c_2 = -5$	$[-7, \infty)$	¾
$c_3 = 11$	$[-1, \infty)$	0

We may see that the range and shadow price using optimal partitioning and the simplex method are same.

4. Concluding Remarks

The results of sensitivity analysis by using the simplex method (using the optimal basis approach) for cases where one of the primal or dual optimal solution is not unique, is not as perfect as the results obtained by using optimal partition approach. When the primal and the dual have a unique optimal solution, simplex method and optimal partition approach give the same information.

References

[1] C. Roos, T. Terlaky and J.-P. Vial, Interior Point Methods for Linear Optimization, New York: Springer, 2006.

[2] B. Jansen, J. de Jong, C. Roos and T. Terlaky, "Sensitivity Analysis in Linear Programming: Just be Careful!," *European Journal of Operations Research,* vol. 101, pp. 15-28, 1997.

APPLICATION OF OPTIMAL CONTROL FOR A BILINEAR STOCHASTIC MODEL IN CELL CYCLE CANCER CHEMOTHERAPY

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Abstract. A model of stochastic bilinear is introduced to study the effect of cancer chemotherapy on the kinetic of the cell cycle disturbed by additive noise. In the design of the chemotherapy or the diseases treatment, it is important to quantify the effect of a drug and to accommodate the random disturbance during treatment period. Optimal control is required to control the effect of chemotherapy because the substance contained in the drug may not only kill the cancer cells but also the normal cells. The normal cells which being exposed to the drug and the other disturbances factor on cell cycles, we assume as the additive noise. The additive noise disturbs the system at random. In this paper to see the numbers of cancer cells during chemotherapy period, will be solved a Riccati equation which the main problem on optimal control theory. Numerical simulation will explain the effect of the drug to number of cancer cells during treatment with random disturbance on each phase of the cell cycle. Particles Swarm Optimization method guess the initial of adjoin equation to solve stochastic optimal control. This methodology enables the authors quantify the effects of chemotherapy and solve bilinear stochastic optimal control.

Key words: bilinear stochastic control, optimal control, white noise, cell-cycles, cancer chemotherapy

1. Introduction

In the design of cancer chemotherapeutic agents it is important to quantify the effect of the drugs to the cancer cells in cell cycles and to accommodate the disturbances during treatment period. This information will enable to improve the effectiveness of chemotherapy treatment. Optimal control is required to control the effect of chemotherapy because the substance contained in the drug may not only kill the cancer cells but also the normal cells.

The cell cycles is divided into four phases : Pre-synthesis phase (G_1), DNA synthesis phase (S), Post-synthesis phase (G_2) and Mitotic Phase (M) in which cell division occurs. Each cell passes through a sequence of phases from cell birth to cell division. Starting point is a growth phase G_1 after which the cell enters a phase

S where DNA synthesis occurs. Then a second growth phase G_2 takes place in which the cell prepares for mitosis or phase M. Here cell division occurs. Depending on the medical aspects taken into account, the cell-cycle is divided into compartments which describe the different cell phases or combine phases of the cell cycle into clusters. The simplest and at the same time most natural models divide the cell cycle into three compartments, respectively. In these models the phases G_2 and M are combined into one compartment as mitosis phase. On the basis of DNA contents, the fractions of cells in the G_1, S, and $G_2 + M$ can be determined. Thus the effects of chemotherapeutic agents on cell cycle progression be studied.

We will see the effect of 1 mg Melphalan on cell cycle chemotherapy. Melphalan, known as Alkeran, is used to treat multiple myeloma, as well as ovarian, breast, and prostate cancer. It is also used for other cancers and sometimes for noncancerous conditions. Melphalan is a member of the general group of chemotherapy drugs known as alkylating agents. It works by interfering with and stopping the growth of cancer cells, which causes them to die.

The research on bilinear control systems applied to cancer chemotherapy was be studied by [1], [4], and [5]. In this research, the bilinear model in [1] will be disturb by additive noise. The additive noise disturbs the system at random and is assumed as a Brownian Motion/Wiener Processes, known as white noise. The early research [1] concluded that Melphalan was most effective in mitosis phase. But, it is not accommodate the others factors we didn't know that affect the cell cycles such as (kidney disease, liver disease (including hepatitis), heart disease, congestive heart failure, diabetes, gout, or infections). In practise, these conditions may impact the medicine to give the more effect in cell cycles. Therefore, we think its important to add the noises in this paper.

In this paper to see the number of cancer cells during chemotherapy period, we use the complete state information. We will solve a Riccati differential equation for bilinear systems which the main problem on optimal control theory. Numerical simulation will explain the effect of the drug to number of cancer cells during treatment with random disturbances on each phase of the cell cycle. Particles Swarm Optimization method guess the initial of adjoin equation to solve stochastic optimal control. This methodology enables the authors quantify the effects of chemotherapy and solve bilinear stochastic optimal control

2. Formulation of The Problem

The dynamics of the cell cycles is represented by the compartmental model shown in Figure 1. This model yields the following bilinear system of differential equation is based on [1]. x_1, x_2, x_3 represent the number of cells at time t in the phases of the cell cycle pre-synthesis G_1, synthesis S, and post-synthesis $G_2 + M$, respectively and $k_1 + u_1, k_2 + u_2, k_3 + u_3$ represent the flow rate parameter of the cell cycles change in time when cells are exposed to drugs, while unperturbed cells maintain constant parameters. From [1] we can define that the constant k_1, k_2, k_3 represent the constant flow rate parameters of unperturbed cells and control input u_1, u_2, u_3 represent the effects of the drug on each phase of the cycle at time t.

We assume that the dynamic of x_1, x_2, x_3 was affected by the random vectors $W_1(t), W_2(t), W_3(t)$ because there are the others factors we didn't know that affect the dynamic of x_1, x_2, x_3 besides the numbers of circulated cells in cell cycle. While, $W_1(t), W_2(t), W_3(t)$ are the random disturbances on systems represented as stochastic processes because on biological dynamic systems evolve under stochastic force.

The most important stochastic process in continuous time is the Wiener process also called Brownian Motion.

$$\dot{x}_1(t) = 1.6(k_3 + u_3(t))x_3(t) - (k_1 + u_1(t))x_1(t) + B_{W1}W_1(t) \quad (1)$$
$$\dot{x}_2(t) = (k_1 + u_1(t))x_1(t) - (k_2 + u_2(t))x_2(t) + B_{W2}W_2(t) \quad (2)$$
$$\dot{x}_3(t) = (k_2 + u_2(t))x_2(t) - (k_3 + u_3(t))x_3(t) + B_{W3}W_3(t) \quad (3)$$

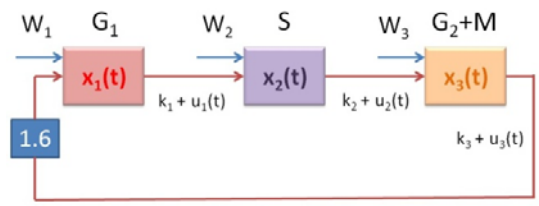

Figure 1. Compartmental Model

Let the vector $X(t) = (x_1(t),x_2(t),x_3(t))^T$, $u(t) = (u_1(t),u_2(t),u_3(t))^T$, and $W(t) = (W_1(t), W_2(t), W_3(t))^T$. So the models (1),(2), and (3) can be written in the form of completely controllable, stochastic system for bilinear :

$$\dot{X}(t) = AX(t) + B_u(t) + \sum_{i=1}^{3} N_i x_i u_i + B_W W(t) \quad (4)$$
$$X(0) = X_0 \quad (5)$$

where,

$$A = \begin{bmatrix} -k_1 & 0 & 1.6k_3 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & -k_3 \end{bmatrix},\ B_u = 0,\ N_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},\ N_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix},$$

$N_3 = \begin{bmatrix} 1.6 \\ 0 \\ -1 \end{bmatrix}$, and B_W is defined as random disturbances weight matrix will be determined when we construct the stochastic optimal control in next section.

The chemotherapy effects can be seen from the numbers of the cells which enter to each phase in cell cycles when cells are exposed to drugs. It is represented by the flow rate parameters $k_1 + u_1$, $k_2 + u_2$, and $k_3 + u_3$. To define the effect of chemotherapy we assume that the combined effects of the drug and the washing cells (cells which were put in a new medium but were not exposed to the drug) use the data for cells exposed to the drug and the values of the constant cell cycle parameters. The constant parameters of the model k_1, k_2, k_3 with u_i for $i = 1,2,3$ was be identified from the unperturbed cells data, i.e. cells which were without washing or being exposed to the drug.

3. Stochastic Optimal Control

3.1. Wiener Processes.

Definition 3.1. *(Brownian Motion) A stochastic process $W(t)$ is called Brownian*

Motion if it satisfies the following condition

(1) *Independence : $W(t + \Delta t) - W(t)$ is independent of $\{W(\tau)\}$ for all $\tau \le t$*
(2) *Stationarity : The distribution of $W(t + \Delta t) - W(t)$ does not depend on t*
(3) *Continuity : $\lim_{\Delta t \to 0} \frac{P(|W(t+\Delta t) - W(t)| \ge \delta)}{\Delta t} = 0$ for all $\delta > 0$*

This definition induces the distribution of the process $W(t)$.

Theorem 3.2. *If $W(t)$ is a Brownian motion, then $W(t) - W(0)$ is a normal random variable with mean μt and variance $\sigma^2 t$, where μ and σ are constant real numbers.*

The random disturbances is assumed as a standard Brownian Motion. We have the definition is as following:

Definition 3.3. *A Brownian Motion is called standard if it satisfies the following condition*

$$\begin{aligned} W(0) &= 0 \\ E[W(t)] &= 0 \\ E[W^2(t)] &= 1 \end{aligned}$$

3.2. The bilinear stochastic optimal control for identifying the drugs effects.

In this section, the following stochastic optimal control problem is considered for a dynamic system with the state vector $X(t)$ □R^3, the admissible control vector $u(t)$ □U□R^3 (where U is a time-invariant, convex, and closed subset of R^3) and the standard vector Brownian Motion $W(t)$ □R^3.

$$\mathrm{d}X(t) = (AX(t) + B_u(t) + {}^{X}N_i x_i u_i)\mathrm{d}t + B_W \mathrm{d}W(t) \quad (6)$$
$$i=1$$

$X(0)=X_0$(7) where A, B_u, N_i for i = 1,2,3 was defined in equation (4) and (5), and B_w is defined as covariance error matrix,

$$B_W = \begin{bmatrix} \sqrt{1.6 b_3 x_3 + b_1 x_1} & 0 & 0 \\ 0 & \sqrt{b_1 x_1 + b_2 x_2} & 0 \\ 0 & 0 & \sqrt{b_2 x_2 + b_3 x_3} \end{bmatrix}$$

with $b_1 = k_1 + u_1(t)$, $b_2 = k_2 + u_2(t)$, and $b_3 = k_3 + u_3(t)$.

In this paper, we consider the stochastic models for cancer chemotherapy with objective function. The objective is to find the control input $u(t)$ which yields the optimal to fit $X(0) = X_0$, $X(t) = X_t$, and $X(t_f) = X_{tf}$ fixed to the data from [1]. The optimal control minimize the objective function is as following:

$$J(X,u) = E\begin{bmatrix} (X(t_f) - X_{t_f})^T P_{t_f} (X(t_f) - X_{t_f}) \\ + \frac{1}{2t_f}\int_0^{t_t} (X(t) - X_t)^T Q (X(t) - X_t) + u(t)^T R u(t) \mathrm{d}t \end{bmatrix} \quad (8)$$

where P_{tf}, Q, R □R with $P_{tf} > 0$, $Q \ge 0$, and $R > 0$ for all t □ $[0, t_f]$. Subject to the constrained equation (6).

We use the following theorem to solve stochastic optimal control for bilinear systems in (6).

Theorem 3.4. *(Stochastic Hamiltonian Jacobi Bellman Theorem) If the partial differential equation*

$$J(X,t) = min_{u\in\mathfrak{U}}\left\{ \begin{array}{c} Ru^Tu + J_x(X,t)(AX + B_u u + \sum_{i=1}^{3} N_i x_i u_i) \\ +\frac{1}{2}tr(J_{xx}(X,t)B_w B_w^T) \end{array} \right\}$$

with the boundary condition $J(X,t_f) = P_{tf}$ *admits a unique solution, the globally optimal state feedback control law is*

$$u = argmin_{u\in\mathfrak{U}}\left\{ \begin{array}{c} Ru^Tu + J_x(X,t)(AX + B_u u + \sum_{i=1}^{3} N_i x_i u_i) \\ +\frac{1}{2}tr(J_{xx}(X,t)B_w B_w^T) \end{array} \right\}$$

Proof. A rigorous proof of this theorem can be found in [2].

We have that optimality conditions satisfies:

$$u_i = -\frac{1}{2R}\left(B(J_x(X,t))_i + J_x^T(X,t)N_i x_i\right) \tag{9}$$

Where, $i = 1,2,3$ and $J_x(X,t) = P(t)X(t)$. With this optimal control law, the Hamiltonian Jacobi Bellman partial differential equation in Theorem (3.4) has the following from

$$J_t + \frac{1}{2}X^TQX - J_xB_uR^{-1}B_u^TJ_x^T + X^TA^TJ_x^T + J_xAX + B_w^TJ_{xx}B_w = 0 \tag{10}$$

To solve the stochastic optimal control, we use a adjoint equation known as Riccati differential equation. It is as following:

$$\begin{aligned} \dot{P} &= \begin{array}{c} [A^TP + PA - PB_uR^{-1}B_u^TP + Q] - \\ R^{-1}[B_u^T + X^T(\textstyle\sum_{i=1}^{3} N_i)^T]PX(\textstyle\sum_{i=1}^{3} N_i)P - \\ PB_uR^{-1}X^T(\textstyle\sum_{i=1}^{3} N_i)^TP - \\ PR^{-1}[B_u^T + X^T(\textstyle\sum_{i=1}^{3} N_i)^T]PX(\textstyle\sum_{i=1}^{3} N_i) \end{array} && (11) \\ P(t_f) &= 0 && (12) \end{aligned}$$

with P is the symmetric and positive definite matrix.

4. Numerical simulation

In the early research by [1] with deterministic optimal control approach, they got that Melphalan given the effective effect in mitosis phase. It was consistent with known data. In this research, we will see the effects of 1 mg Melphalan on cell cycle with stochastic optimal control approach. The optimal control is accounted through the Riccati differential equation (11). The chemotherapy effects is determined from the cells traverse rate $k_1 + u_1(t)$, $k_2 + u_2(t)$, and $k_3 + u_3(t)$.

We have to solve stochastic optimal control in equation (9) which minimize the objective function J subject to the systems (6) and (11). The procedure to solve the optimization problem is based on the Particle Swarm Optimization algorithm. PSO algorithm is used to solve optimal control so that the objective function (8) minimized. The constant parameters k_1, k_2, k_3 was determined from unperturbed data where $k_1 = 0.050/hour$, $k_2 = 0.087/hour$, and $k_3 = 0.159/hour$. In next explanation, we will quantify the effects of the chemotherapy as the flow rate $k_1 + u_1(t)$, $k_2 + u_2(t)$, and $k_3 + u_3(t)$ with $u_i(t)$ is dependent time.

For this problem, PSO algorithm to solve optimal control is as following

(1) Create an agent population (particles) as solution of initial adjoin equation.
Define $\tilde{x}^0_j$ and v_j^0 with j is the numbers of agents population.
(2) Solve the systems with these result.
(3) Evaluate the objective function $J(x,\tilde{u})$.
(4) Define the best particle position, i.e. particle that has an optimal value.
(5) Update particle's velocities

$$v_j^{t+1} = v_j^t + \varphi_1 c_1^t(\text{pbest}_j^t - \tilde{x}_j^t) + \varphi_2 c_2^t(\text{gbest}_j^t - \tilde{x}_j^t)$$

where v_j^t is particle's inertia, $\varphi_1 c_1^t(\text{pbest}_j^t - \tilde{x}_j^t)$ is personal influences, and $\varphi_2 c_2^t(\text{gbest}_j^t - \tilde{x}_j^t)$ is social influences.
(6) Update particle's

$$\tilde{x}_j^{t+1} = \tilde{x}_j^t + v_j^{t+1}$$

(7) Go to 2nd step unless some termination criteria are met.

Figure 2 shows the result for relative numbers of cells on unperturbed cells. The results of the determination of the constant parameters k_1,k_2,k_3 and the graph shows the fit of the data [1]. We use PSO algorithm to fit simultaneously the three set of data for the contents of G_1, S, and $G_2 + M$. For stochastic model (the blue one) we see that he numbers of cells in G_1, S, and $G_2 + M$ increase allow the time (in hour). It's mean that, the normal cells still grow and die in controlled (normal) way.

Laboratory data in Figure 2 are shown by the green dot one. Objective function for this problem has $Q = 0$, but we expect the final point of state close to the data. Thus, we can see the original dynamics of bilinear models. The effects of either washing the cells or the effects of the drugs the sum of the value for the control functions and the parameters, i.e. $k_1+u_1, k_2+u_2, k_3+u_3$ must be considered.

Figure 3 shows the average result of cells traverse rate as effect of the drugs in 5 hours for $k_1 + u_1, k_2 + u_2, k_3 + u_3$ in the case of washing the cells. $k_1 + u_1$ represent the cells traverse from G_1 (pre-synthesis) phase to S (synthesis) phase, $k_2 + u_2$ represent the cells traverse from S (synthesis) phase to $G_2 + M$ (mitosis) phase, and $k_3 + u_3$ represent the cells traverse from $G_2 + M$ (mitosis) phase back to G_1 (pre-synthesis) phase. In Figure 3 we can see that $k_1 + u_1$ for stochastic model is greater enough than $k_1 + u_1$ for deterministic model. Both of them toward to 0.05*/hour* in 40 hours. Then, $k_2 + u_2$ both of stochastic and deterministic model closer together. Its mean that deterministic and stochastic model have same control effect in synthesis phase. And $k_3 + u_3$ both of them stable and toward to 0.1597*/hour* in 40 hours.

Figure 4 shows the numerical simulation result for cell numbers of washed cells. The numbers of normal cells in washed cells for stochastic model relatively increase. This results are consistent enough with the known experimental data. Figure 5 is the result of the drug effects for exposed cells. the sum of the value for the control functions and the parameters, i.e. $k_1 + u_1, k_2 + u_2,$ and $k_3 + u_3$ must be considered for exposed cells. The effects was be accounted on average of 10 hours due to the period effects of exposed cells. Figure 6 shows cells relative numbers for cells exposed to the drug 1 mg Melphalan.

Comparing the figures for the different experiments, we observe the following:

(1) In the first period of the washed cells experiment, following the exposure to the drug Melphalan, there is a enough increasing in the parameter $k_1 + u_1$, which corresponds to cell traverse from G_1 to S phase. Then, there is decreasing in the parameter $k_2 + u_2$, corresponds to cell traverse from S to

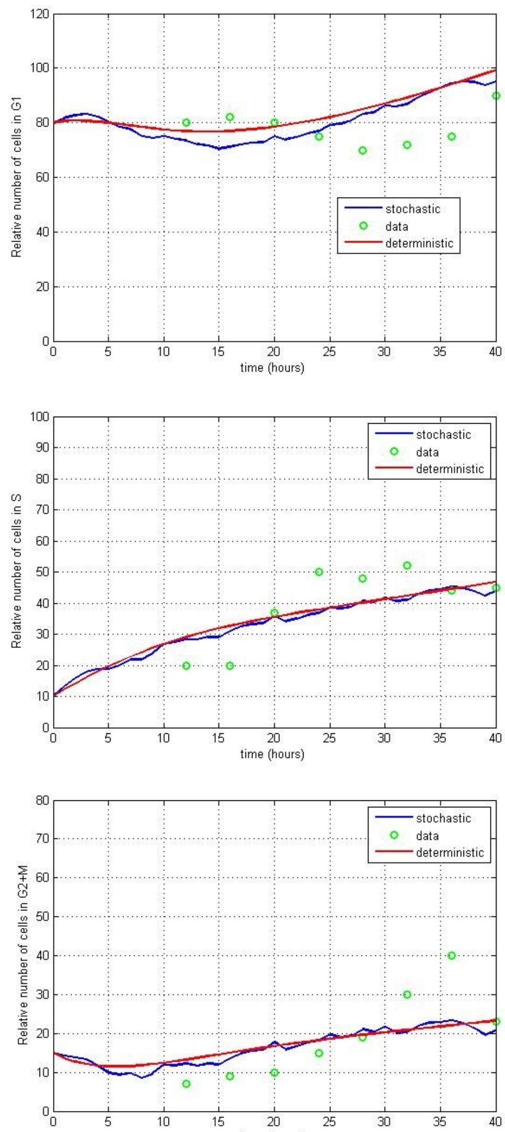

Figure 2. Relative numbers of unperturb cells

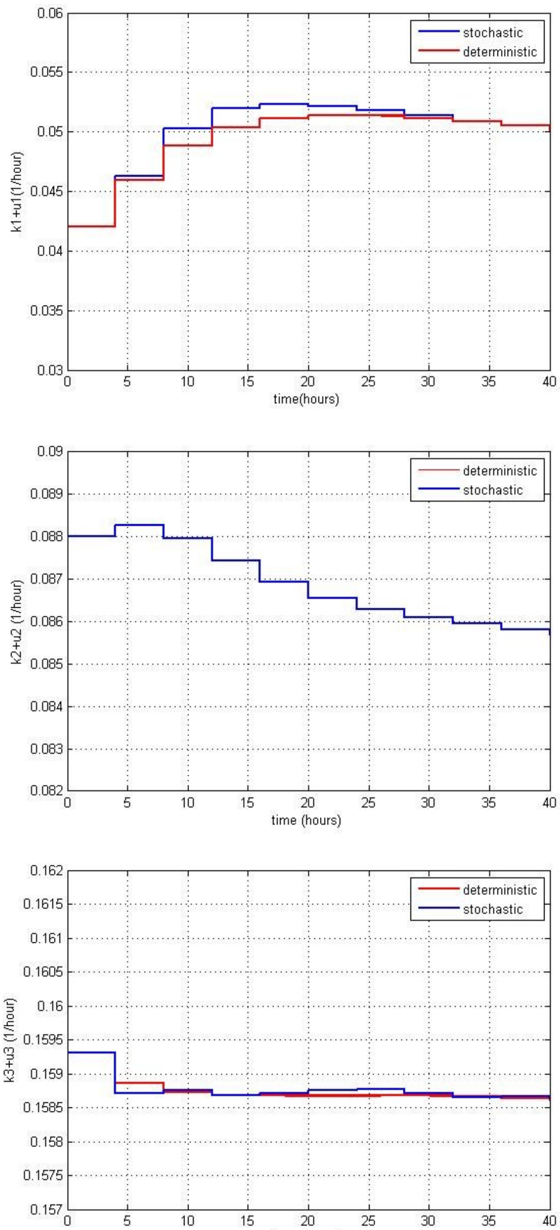

Figure 3. The effects of the drugs for washed cells

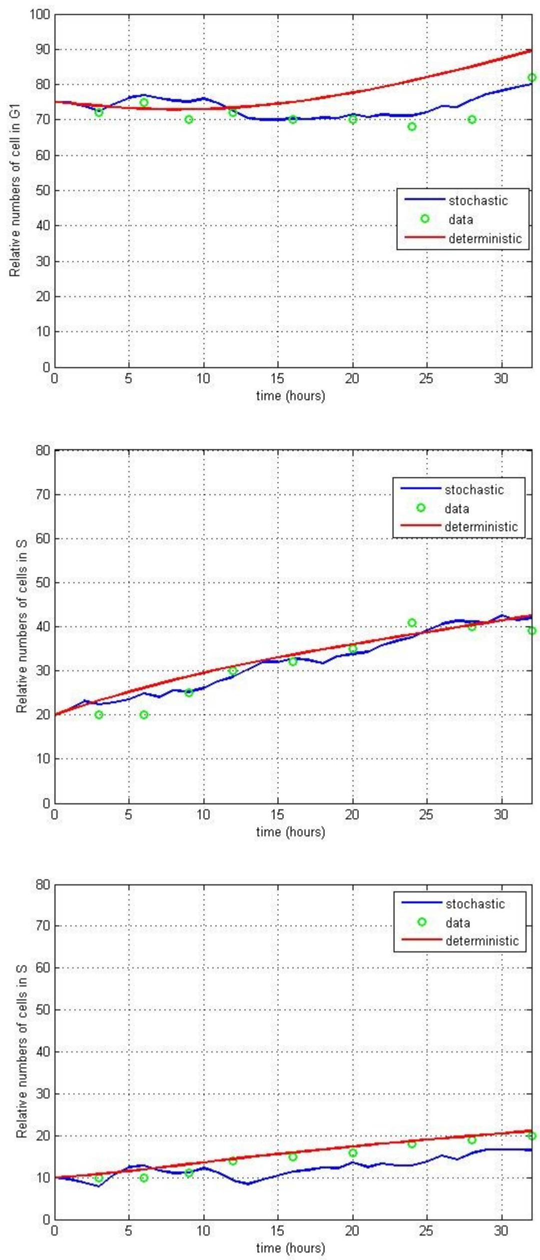

Figure 4. Relative numbers of washed cells

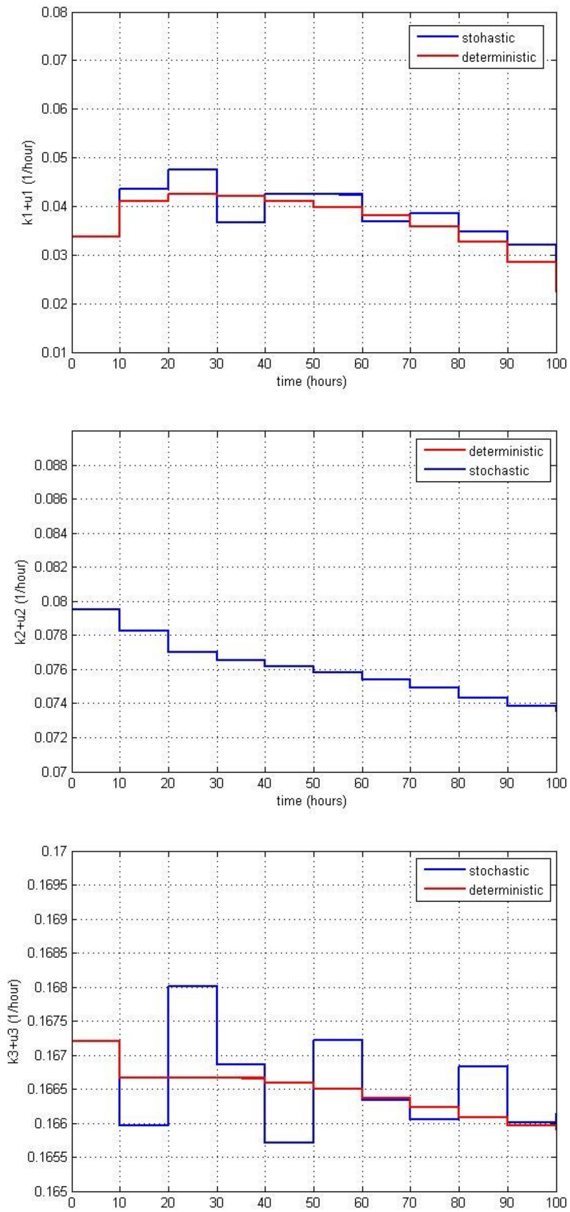

Figure 5. The effects of the drugs for exposed cells

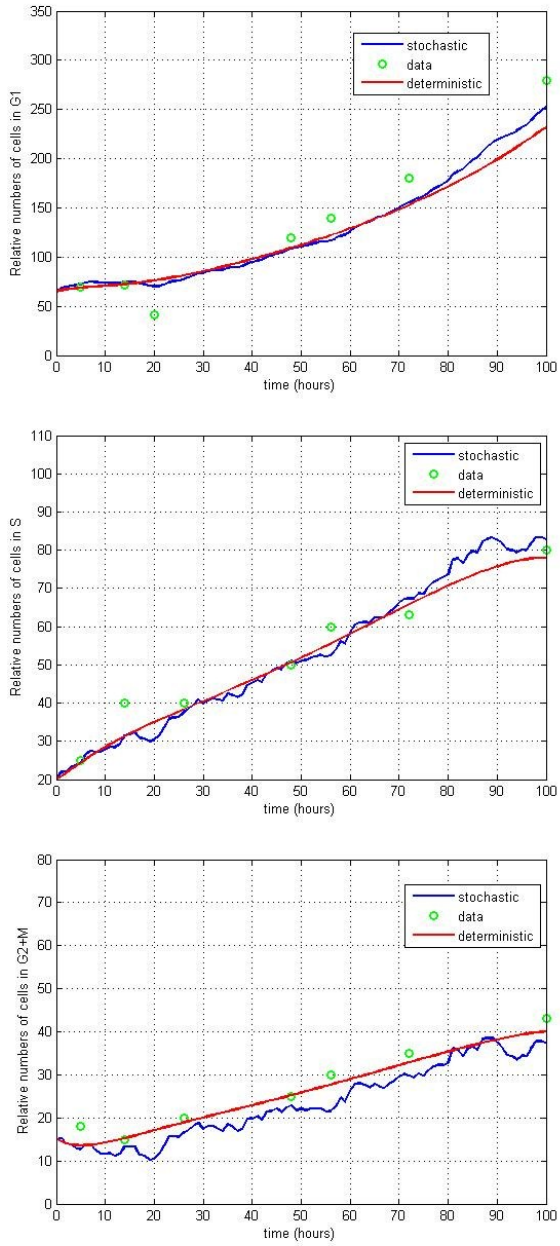

Figure 6. Relative numbers of exposed cells $G_2 + M$. This effect was prolonged as the drug dosage was same in 1mg Melphalan (Figure 3a and Figure 5a). In same doses tested, this parameter returned to the original condition of the unperturbed cells within 40 hours from the start of the experiment.

(2) In all experiments, exposure to the drug resulted is stable and closer together between deterministic and stochastic model in the parameter k_2+u_2 which means cell traverse from S phase to $G_2 + M$ phase (can be seen at Figure 3b and Figure 5b).

(3) Cell traverse from synthesis S to the mitosis of the cycle $G_2 + M$ which is reflected by $k_3 + u_3$ values, is affected by the drug in a dose related fashion. The early research [1] concluded that Melphalan was the most effective on cells on mitosis where the parameter $k_3 + u_3$ dropped to a low constant level. While in this research, the control in which cells were sustained in a medium being exposed to the drug, the parameter $k_3 + u_3$ is random enough. In the case where cells were exposed to the smallest dosage (1 mg Melphalan) there is an initial transient sometimes increase and decrease in the parameter rapidly. Therefore, the random disturbances

5. Concluding Remarks

This simulation result show that with stochastic optimal control we get that 1 mg Melphalan give random effect to parameter $k_3 + u_3(t)$ on mitosis phase $G_2 + M$ of exposed cells, whereas the early research by [1] concluded that Melphalan given the most effective on mitosis phase because the rate of cell traverse $k_3 + u_3$ decreased along treatment period. With stochastic optimal control we can capture the phenomena effect of the disturbances factor on cell cycles chemotherapy. On the mitosis phase, stochastic result conclude that Melphalan is not effective enough on mitosis phase. According to washed and exposed cells stochastic model simulation, the results are consistent enough with the experimental data. There is no significant different between the effects in synthesis phase for washed cells and exposed cells. Thus, stochastic result conclude that 1mg Melphalan is effective on cells at synthesis phase and suggest that the optimal control approach provides a quantitative method for determining drug effects on cell cycles.

References

[1] Yael Biran and Bayliss McInnis, Optimal Control of Bilinear Systems: Time Varying Effects of Cancer Drugs, Automatic vol.15 pp. 325-329, 1979

[2] A. Chinchuluun et al. (eds.), Optimization and Optimal Control, Springer Optimization and Its Applications 39, DOI 10.1007/978-0-387-89496-6 18, Springer Science+Business Media, LLC, 2010

[3] Suzanne Lenhart and John T. Workman, Optimal Control Applied to Biological Models, Chapman and Hall/CRC, 2007

[4] Urzula Ledzewicz and Heinz Schattler, Analysis of a cell cycle specific model for cancer chemotherapy, J. of Biological Systems, 10, pp. 183-206, 2005

[5] Roberd Saragih, Ednawati Rainarli, Aplikasi kontrol bilinier pada sumsum tulang dengan kemoterapi cell-cycle spesifik, Prosiding KNM XIII 793-800, 2006

OUTPUT TRACKING OF SOME CLASS NON-MINIMUM PHASE NONLINEAR SYSTEMS

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Abstract. In this paper, we will design an input control to track the output of a non-minimum phase nonlinear system. The design of the input control is based on an input-output linearization method and gradient descent control. To perform the design of the input control, the other output should be selected such that the systems becomes minimum phase systems. Furthermore, the desired output of the output which has been selected will be set based on the desired output of the original system.

Key words and Phrases: input-output linearization, steepest descent control, minimum phase system,non-minimum phase system.

1. Introduction

A system called minimum phase systems if the origin of the zero dynamics is asymptotically stable, but if zero dynamics is unstable, then the system called non-minimum phase [6] . In [1], Isidori has shown that for the minimum phase systems, then it can be choose a static control law such that the output of the system goes to zero while keeping the state of the system bounded locally. While that in [2] has introduced a dynamic feedback control which is a modification of the steepest descent control. The modified steepest descent control success to handle the problem that arise if relative degree of the nonlinear system not well defined. Recently, output tracking problems on nonlinear non-minimum phase systems have been investigated intensively. The stable inversion proposed in [3], [4] is an iterative solution to the tracking problem with the unstable zero dynamics. This method requires the system to have well defined relative degree and hyperbolic dynamics, i.e. no eigenvalues on imaginary axis. In [5], proposed the control design procedure for the output tracking. The design procedure consists of two steps. In the first step, the standard input output linearization is applied. In the second step, we group a subset of the output with the internal dynamics as one subsystems, which is usually nonlinear, and the rest of output as the other subsystem which is linear, the nonlinear subsystems is linearized about its equilibrium. In [7], Riccardo Marino and Patrizio Tomei have shown how to design a globally stabilizing dynamic output feedback controller of order $n + 2(\rho - 1)$(n is the system order, ρ is the relative degree) for a class of nonlinear nonminimum phase systems. The

system are required to be minimum phase with respect to a linear combination of the state variables. In [8], the asymptotic output tracking which is a class of causal nonminimum phase uncertain nonlinear systems is achieved by using higher order sliding modes (HOSM) without reduction of the input-output dynamics order. In [9], a new nonlinear dynamic controller is described based on the gradient descent control. Performance index is generated by error of output system from output desired value and internal state of the system. adding of an internal state to maintain the stability of internal dynamic of the system.
In this paper, we will design the input control which ensures that the nonlinear system is stable asymptotically. If the system has relative degree well defined, we used the input output linearization method [1] to design input control. Then if the relative degree of the system is not well defined, to design of the input control based on the modification of the steepest descent control. Modification is the addition of an input artivisial of the steepest descent control. Furthermore, the desired output of the output which has been selected will be set based on the desired output of the original system.

2. Output Tracking

We will investigate the output tracking for a non-minimum phase nonlinear system. The non-minimum phase system in the following form :

$$\dot{x} = \mathbf{A}x + bu + \varphi(y),\ x \in \mathbf{R}^n,\ u \in \mathbf{R} \qquad (1)$$

$$y = x_1 \qquad (2)$$

in which $\varphi(y)$ is a smooth vector field in $\mathbf{R}^n$ with $\varphi(0) = 0$, $b = [0,\dots,0,b_r,\dots,b_n]^T$ with $b_r \neq 0$,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Our objective is to make the output system (1)-(2) tracks the desired output while keeping the state bounded. To keep the state bounded is difficult for non-minimum phase system. In this paper, we design a controller such that the output system (1)-(2) tracks the desired output while keeping the state bounded. To perform the design of the input control, the other output should be selected such that the system (1) becomes minimum phase with respect to a new output. Thus, in this paper we assume that

Assumption 1. *There exists a linear combination of the state variables $\mu = t_1 x$ such that the zero-dynamics (1)-(2) is asymptotically stable.*

We consider the system (1)-(2). Based on assumption 1, consider now a new output

$z_1 = \mu = t_1 x = (t_{11},\dots,t_{1n})x$

with the relative degree of system (1) with respect to μ is still equal to r. The system (1) can be transformed to

$\dot{z}_1 = z_2\ \dot{z}_2 = z_3$

$$\vdots \tag{3}$$

$\dot{z}_r = b(z) + a(z)u,\ \dot{\eta} =$

$q(z)\ y = z_1.$

Before applying the control law, we have to set up the output desired for z_1, i.e. z_{1d}. In this paper we consider the system which satisfies the following assumption

Assumption 2. *$\lambda_l(x) = x_l$, $l \in \{2,...,n\}$ then $\dot{x}_l = f_l(x_l,x_1)$ can be solved by substituting $x_1 = y_d(t)$*

Thus, $\lambda_{ld} = x_l(t)$.
Based on assumption 2, we obtain

$z1d = t11yd + t12\lambda 2d + \cdots + t1n\lambda nd$

let $e_1 = z_1 - z_{1d},\ e_2 = \dot{z}_1 - \dot{z}_{1d},\ \cdots,\ e_r = z_1^{(r-1)} - z_{1d}^{(r-1)}$. Thus

$$\begin{aligned} \dot{e}_k &= e_{k+1},\ 1,...,r-1 \\ \dot{e}_r &= b(e+z_d,\eta) + a(e+z_d,\eta)u - z_{1d}^{(r)}, \\ \dot{\eta} &= q(e+z_d,\eta,\eta). \end{aligned} \tag{4}$$

By choosing

$$u = \frac{1}{a(e+z_d,\eta)}(-b(e+z_d,\eta) + z_{1d}^{(r)} - \sum_{i=1}^{r} c_{(i-1)}(e_1^{(i-1)})) \tag{5}$$

where $c_0, c_1, ..., c_{r-1}$ are real numbers,

$e(t) = col(e_1(t).\dots,e_r(t))$ *and* $z_d(t) = col(z_{1d}(t), z_{1d}^{(1)}(t), \dots, z_{1d}^{(r-1)}(t))$

Then system (4) can be written as

$$\dot{e} = \mathbf{A}e, \tag{6}$$

$$\dot{\eta} = q(e+z_d,\eta), \tag{7}$$

where the matriks **A** has a characteristic polynomial : $p(s) = c_0 + c_1 s + \cdots + cr{-}1sr{-}1 + sr$.

By choosing the value of $c_i; i = 0,...,r$ such that all the roots oh the polynomial $p(s)$ have negative real part, and applying the control law (5), then the equilibrium point $(e,\eta) = (0,0)$ of the system (4) is asymptotically stable. (see Proposition 4.5.1 in [1]).

Thus e_1 tend to zero if time t goes to infinity. Then z_1 tend to z_{1d} if time t goes to infinity. Thus x_1 tracks to the desired output $y_d(t)$.

Next, if the relative degree of the system (1)-(2) is not well defined.
We construct the performance index as a descent function as follow :

$$F(z_1, \dot{z}_1, \dots, z_1^{(r)}(t)) = \left(\sum_{j=0}^{r} a_j (z_1 - z_{1d})^{(j)}\right)^2. \tag{8}$$

By"Trajectory Following Method" [10], the control u is determined from the differential equation

$$\dot{u} = -\frac{\partial F}{\partial u} = -2a_r \left(\sum_{j=0}^{r} a_j (z_1 - z_{1d})^{(j)} \right) \frac{\partial (z_1 - z_{1d})^{(r)}}{\partial u}, \qquad (9)$$

where the control law in (9) is called the steepest descent control [2]. Furthermore calculate the time derivative of descent function (8) along the trajectory of the extended system

$$\dot{z}_k = z_{k+1}, \; k = 1, \ldots, r-1 \qquad (10)$$

$$\dot{z}_r = a(z\,,\,\eta) + b(z\,,\,\eta)u \qquad (11)$$

$$\dot{\eta} = q(z\,,\,\eta) \qquad (12)$$

$$\dot{u} = -2a_r \left(\sum_{j=0}^{r} a_j (z_1 - z_{1d})^{(j)} \right) \frac{\partial (z_1 - z_{1d})^{(r)}}{\partial u}. \qquad (13)$$

Now, we have

$$\dot{F}(y, \dot{y}, \ldots, y^{(r)}) = \left(\frac{\partial F}{\partial z} \dot{z} + \frac{\partial F}{\partial \eta} \dot{\eta} + \frac{\partial F}{\partial u} \dot{u} \right). \qquad (14)$$

From equation (14), we see that the value of time derivative of the descent function along the trajectory of the extended system can not be guaranteed to be less than zero for $t \geq 0$.

Now we modify the steepest descent control (9) by adding an artificial input v.

Then the extended system (1) becomes

$$\dot{x} = \mathbf{A}x + bu + \varphi(y), \; x \,\square\, \mathbf{R}^n, \; u \,\square\, \mathbf{R} \qquad (15)$$

$$\dot{u} = -\frac{\partial F}{\partial u} + v. \qquad (16)$$

From equation (14), we have

$$\dot{F}(y, \dot{y}, \ldots, y^{(r)}) = \left(\frac{\partial F}{\partial x} \dot{x} + \frac{\partial F}{\partial u} \dot{u} \right) = \frac{\partial F}{\partial x} \dot{x} + \frac{\partial F}{\partial u} \left(-\frac{\partial F}{\partial u} + v \right). \qquad (17)$$

From Sontag formula, we get

$$v = \frac{1}{\frac{\partial F}{\partial u}} \left(-\frac{\partial F}{\partial x} \dot{x} - \sqrt{\left(\frac{\partial F}{\partial x} \dot{x} \right)^2 + \left(\frac{\partial F}{\partial u} \right)^2} \right). \qquad (18)$$

The control law in equation (16) is called as modified steepest descent control.

Based on the modified steepest descent control, then $\dot{F}(y, \dot{y}, \ldots, y^{(r)}) < 0$, if $\left(\sum_{j=0}^{r} a_j (z_1 - z_{1d})^{(j)} \right) \neq 0$. Thus, if we choose a_j such that the polynomial $p(s) = a_0 + a_1 s + \cdots + a_{r-1} s^{r-1} + s^r$ is Hurwitz, z_1 tend to z_{1d} if time t goes to infinity. Thus x_1 tracks to the desired output $y_d(t)$.

Example 1. Consider the nonlinear system (SISO)

$$\dot{x}_1 = x_2 + 2x_1^2$$

$$\dot{x}_2 = x_3 - u + 2x_1^2 \qquad (19)$$

$$\dot{x}_3 = u - 4x_1^2$$

$$y = x_1, \; y_d = \sin(t).$$

The nonlinear system (19) has relative degree 2 at any point x_0 (relative degree of the system is well defined). In normal form, the nonlinear system (19) becomes

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= 4z_1^2 - z_2 + 4z_1z_2 + \eta - u \\ \dot{\eta} &= \eta - z_2. \end{aligned} \qquad (20)$$

Because the stability of zero dynamics is unstable, the nonlinear system (19) is the non-minimum phase. Now, redefining output $z_1 = \mu = x_1 + 2x_2 + 2x_3$. By considering the new output, the relative of the system (19) is 2 at any point x_0
and normal form

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= b(z) + a(z)u \\ \dot{\eta} &= -\eta + z_2, \end{aligned} \qquad (21)$$

where $b(z) = x_3 - 6x_1^2 - 4x_1x_2 - 8x_1^3$, $a(z) = 1$. The zero-dynamics of the system (19) are asymptotical stable. Thus the system (19) is the minimum phase with

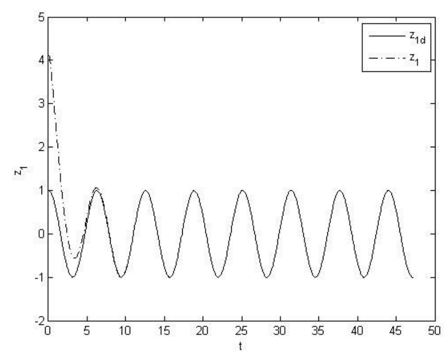

Figure 1 : Output tracking z_1 *to* z_{1d}

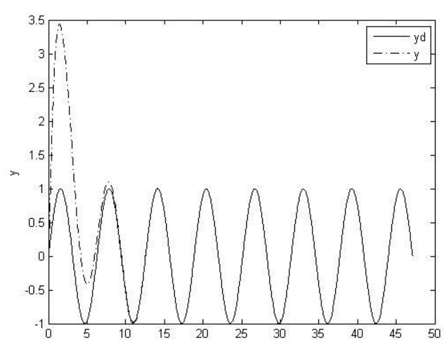

Figure 2: Output tracking (original system) y *to* y_d

respect to a new output.

Let $y_d(t) = sin(t) = x_{1d}(t)$. Next, we chose $z_{1d}(t)$ such that if $z_1(t)$ tracks $z_{1d}(t)$, then $y(t)$ tracks to the desired output $y_d(t)$. By replacing x_1 with $x_{1d}(t) = sin(t)$, then we have $x_{2d} = cos(t) - 2sin^2t$. By replacing x_2 with $x_{2d}(t)$, we have a differential equation $\dot{x}_3 - x_3 = sin(t) + 2sin(2t) - 2sin^2t$.

Thus $x_{3d} = -1/2cos(t) - 1/2sin(t) - 2cos^2t + 2$. Next, $z_{1d} = x_{1d} + 2x_{2d} + 2x_{3d} = cos(t)$. According to (5), the input control is

$$u = -b(z) + \ddot{z}_{1d} - c_0(z_1 - z_{1d}) - c_1(\dot{z}_1 - \dot{z}_{1d})$$

Simulation results are shown in Figure 1 and in Figure 2 for constants: $c_0 = 6$, $c_1 = 10$. Initial value: $x_1(0) = 0$, $x_2(0) = 1$, $x_3(0) = 1$.

In Figure 1, the output which has been selected such that the system become minimum phase track the desired output z_{1d}.

In Figure 2, the output of the original system track the desired output y_d.

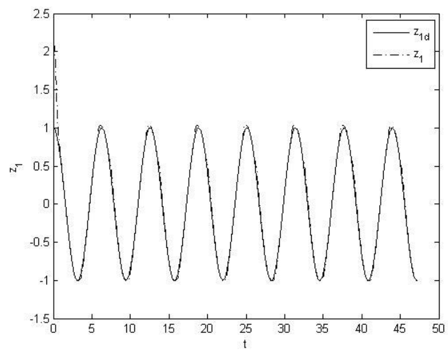

Figure 3: Output tracking z_1 *to* z_{1d}

Example 2.

$$\begin{aligned} \dot{x}_1 &= x_2 + 2x_1^2 \\ \dot{x}_2 &= x_3 - x_2u - 2x_1^2 \qquad (22) \\ \dot{x}_3 &= x_2u \\ y &= x_1,\ y_d = sin(t). \end{aligned}$$

The nonlinear system (22) has relative degree 2 at any point $x_0 \neq 0$ (relative degree of the system is not well defined). The system (22) is the non-minimum phase. By considering the new output $z_1 = \mu = x_1 + 2x_2 + 2x_3$, the zero dynamic are $\dot{\eta} = -\eta$. Therefore the system (22) is minimum phase with respect to the new output. By the same method as in example 1, obtained $z_{1d} = x_{1d} + 2x_{2d} + 2x_{3d} = cos(t)$.

According to (9), the modified steepest descent control is $\dot{u} = -2x_2a_2(a_0(z_1 - z_{1d}) + a_1(\dot{z}_1 - \dot{z}_{1d}) + a_2(\ddot{z}_1 - \ddot{z}_{1d})) + v$, (23) with v as in (18). Simulation results are shown in Figure 3 and in Figure 4 for constants: $a_0 = 35$, $a_1 = 12$, $a_2 = 1$. Initial value: $x_1(0) = 0$, $x_2(0) = 1$, $x_3(0) = 1$, $u(0) = 1$.

In Figure 3, By modified steepest descent control, the the output which has been selected such that the system become minimum phase track the desired output z_{1d}. In Figure 4, the output of the original system track the desired output y_d.

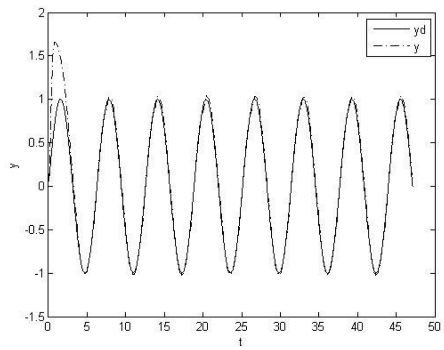

Figure 4: Output tracking (original system) *y to* y_d

3. Conclusions

In this paper, we have investigated the output tracking for a class of nonlinear non-minimum phase system (1)-(2). The input control has been designed for the output tracking. To perform the design of the input control, the systems (1) are required to be minimum phase with respect to a new output, where the new output is a linear combination of the state variables. Furthermore, the desired new output will be set based on the desired output of the original. If the system (1) has relative degree well defined with respect to the new output, we used the input output linearization method to design the input control. Then if the relative degree of the system (1) is not well defined with respect to the new output , to design of the input control based on the modification of steepest descent control.

References

[1] A. Isidori.*Nonlinear Control Systems: An Introduction*. Secon Edition Springer-Verlag Berlin, Heidelberg 1989.

[2] J Naiborhu.*Trajectory Following Method on Output Regulation of Affine Nonlinear Control Systems with Relative Degree not Well Defined*. ITB J.Sci.,Vol. 43A,No.1,2011,73-84.

[3] D.Chen and B.Paden (1996), Stable inversion of nonlinear non-minimum Phase systems, *Int.J.Control* , Vol.64,No.1, 45-54.

[4] D.Chen (1993), Iterative solution to stable inversion of nonlinear non-minimum Phase systems, *Proc. American Control Conference*, June, 2960-2964.

[5] Dong Li. (2005), Output Tracking of Nonlinear Nonminimum Phase Systems : an Enginering Solution, *Proceedings of the 44th IEEE Conference on Decision and control, and the European Control Conference*, Sevilla,Spain,Dec 12-15, 3462-3467.

[6] Hassan K.Khalil. (2002), Nonlinear Systems, *Prentice Hall, New Jersey* , Third Edition.

[7] Riccardo Marino, Patrizio Tomei. (2004), A class of Globally Output Feedback Stabilizable Nonlinear Non-minimum Phase Systems, *43rd IEEE Conference on Decision and Control*, Atlantis, Paradise Island,Bahamas Des 14-17, 4909-4914.

[8] S. Baev, Y.Shtessel, I. Shkolnikov. (2007), HOSM driven outputvtracking in the nonminimum-phase causal nonlinear Systems, *Proceeding of the 46th IEEE Conference on Decision and Control*, New Orleands,LA,USA, Des 12-14, 3715-3720.

[9] J. Naiborhu.*Output Tracking of Nonlinear Non-minimum Phase Systems by Gradient Descent Control*. proceeding of the IASTED International Conference Identification, Control, and Applications (ICA 2009),August 17-19, 2009, Honolulu Hawaii, USA, 110-115

[10] Vincent,T.L. and W.J. Grantham,*Nonlinear and Optimal Control Systems*, John Wiley and Sons, Inc., New York, 1997.

AN ANALYSIS OF A DUAL RECIROCITY BOUNDARY ELEMENT METHOD

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Abstract. Infiltration from irrigation channels is governed by Richard's equation. This equation may not be solved analytically or numerically. To study the infiltration more conveniently, the governing equation is transformed to a Helmholtz equation, which may be solved numerically. A numerical method that may be employed to solve the Helmholtz equation is the dual reciprocity boundary element method (DRBEM). In this study, we employ the DRBEM, to solve a problem involving infiltration from periodic flat channels. The DRBEM is implemented on MATLAB. Using the analytic solution of this problem obtained by Batu, an analysis of the method is made.

Key words and Phrases:Dual reciprocity boundary element method, infiltration, periodic channels, homogeneous soil

1. Introduction

In the last few decades, the problem of water infiltration has been studied by a number of researchers such as Gardner [12], Philip [14, 15], Batu [7], and Basha [6]. Gardner proposed an exponential relationship between the hydraulic conductivity and the suction potential. Philip wrote an excellent review of soil water physics, he also investigated infiltration from spherical cavities, and obtained the exact solution using the method of separation variables. A problem involving steady infiltration from periodic flat channels was studied by Batu. Batu obtained the analytic solution of the problem using the same method as Philip. Basha employed Green's function to study infiltration toward a shallow water table.

The studies carried out by some of these researchers were limited, since the methods used may not be applied to solve infiltration problems with arbitrary geometry of the boundary or boundary conditions. A possible way to deal with such problems is to use numerical methods. A numerical method used to solve such problems was the boundary element method (BEM). This was used by Pullan and Collins [16], Pullan [17], Azis et al [5], Lobo et al [13], Clements et al [10], and Clements and Lobo [11].

The method has successfully handled various infiltration problems. However, this method is only used for special governing equations that can be transformed to a type of Helmholtz equation. This is due to the requirement of the fundamental solution of the Helmhotz equation in the BEM formulation. If the governing equation cannot be transformed to the type of Helmholtz equation, the BEM may not be used.

A method that may be employed to solve more general Helmholtz equations is the dual reciprocity boundary element method (DRBEM). This method requires only the fundamental solution of the Laplace equation. However, to apply this method a number of constant line segments and a number of collocation points must be constructed. These two numbers must be chosen in such a way, so that some degree of accuracy and optimum computational time are achieved.

In this paper, a problem involving infiltration from periodic flat channels is solved numerically using the DRBEM. Different sets of constant line segments and collocation points are tested to obtain numerical solutions. The corresponding analytic solution is used to analyze error obtained from these different sets.

2. Problem Formulation

We consider an array of equally spaced identical flat channels, each of width of $2L$. The distance between two consecutive channels is $2D$. We assume that the channels are sufficiently long and that there is a large number of such channels. Hence the flow pattern is two-dimensional and the influence of outer channels is negligible. The fluxes on the channels and on the soil surface outside the channels are v_0 and 0 respectively. The geometries described are as shown in Figure 1.

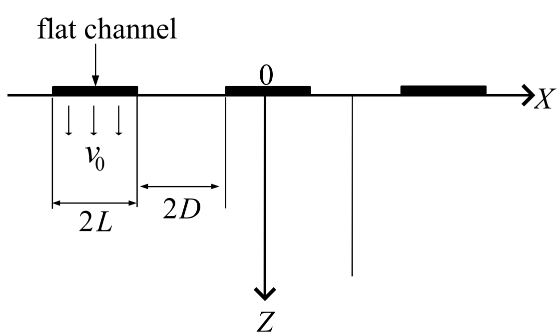


Figure 1. Cross-section of periodic flat channels.

Due to the symmetry of the problem, it is sufficient to consider the semiinfinite region defined by $0 \leq X \leq L + D$ and $0 \leq Z \leq \infty$. This region is denoted by R with boundary C. Along the surface of the channel, the boundary is symbolized by C_1 and along the surface of the soil outside the channel by C_2. The fluxes across C_1 and C_2 are v_0 and 0 respectively. The boundary along $X = L+D$ and $X = 0$, denoted by C_3 and C_4, have zero fluxes.

3. Basic equation

Steady infiltration in porous media is governed by the following Richard's equation

$$\frac{\partial}{\partial X}\left(K\frac{\partial \psi}{\partial X}\right)+\frac{\partial}{\partial X}\left(K\frac{\partial \psi}{\partial Z}\right)=\frac{\partial K}{\partial Z}, \tag{1}$$

where K is the hydraulic conductivity, Z is the vertical physical space coordinate, pointing positively downward [17, 18, 19].

The Kirchhoff transformation

$$\Theta=\int_{-\infty}^{\psi} K\,d\psi,$$

where Θ is the matric flux potential, and the exponential relation

$$K(\psi)=K_s e^{\alpha\psi}, \qquad \alpha>0, \qquad \psi\le 0,$$

where K_s is the saturated hydraulic conductivity and α is an empirical parameter [7, 17, 18, 19], transform equation (1) to the linear equation,

$$\frac{\partial^2\Theta}{\partial X^2}+\frac{\partial^2\Theta}{\partial Z^2}=\alpha\frac{\partial \Theta}{\partial Z}. \tag{2}$$

The horizontal and vertical components of the flux, which are functions of the matric flux potential, are

$$U=-\frac{\partial \Theta}{\partial X},$$

and

$$V=\alpha\Theta-\frac{\partial \Theta}{\partial Z}$$

respectively [7]. The flux normal to the surface with outward pointing normal, $\mathbf{n}=(n_1,n_2)$, is given by

$$F=-\frac{\partial \Theta}{\partial X}n_1+\left(\alpha\Theta-\frac{\partial \Theta}{\partial Z}\right)n_2.$$

Using the dimensionless variables

$$x=\frac{\alpha}{2}X, \quad z=\frac{\alpha}{2}Z, \quad \Phi=\frac{\pi\Theta}{v_0 L},$$

$$u=\frac{2\pi}{v_0\alpha L}U, \quad v=\frac{2\pi}{v_0\alpha L}V, \text{ and } f=\frac{2\pi}{v_0\alpha L}F$$

and the substitution

$$\Phi=\phi e^{z}, \tag{3}$$

in equation (2), we obtain

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=\phi, \tag{4}$$

which is a type of Helmholtz equation. The dimensionless flux is

$$f=-e^{z}\left(\frac{\partial \phi}{\partial n}-\phi\, n_2\right),$$

which yields

$$\frac{\partial \phi}{\partial n}=\phi\, n_2-e^{-z}f.$$

(5)

Fluxes at the surface of the channels are v_0, and therefore dimensionless fluxes are $2\pi/\alpha L$. This implies

$$\frac{\partial \phi}{\partial n} = \frac{2\pi}{\alpha L} e^{-z} + \phi\, n_2, \text{ for } 0 \leq x \leq \frac{\alpha L}{2} \text{ and } z = 0. \quad (6)$$

The condition of zero flux across the soil surface outside the channels, and along $X = 0$ and $X = L + D$ implies

$$\frac{\partial \phi}{\partial n} = -\phi, \text{ for } \frac{\alpha L}{2} \leq x \leq \frac{\alpha}{2}(L + D) \text{ and } z = 0, \quad (7)$$

$$\frac{\partial \phi}{\partial n} = 0, \text{ for } x = 0 \text{ and } z \geq 0, \quad (8)$$

and

$$\frac{\partial \phi}{\partial n} = 0, \text{ for } x = \frac{\alpha}{2}(L + D) \text{ and } z \geq 0. \quad (9)$$

It is assumed that as $X^2 + Z^2 \to \infty$, $\partial\Theta/\partial X = 0$ and $\partial\Theta/\partial Z = 0$ [7]. Thus

$$f = 2\phi\, e^{z}. \quad (10)$$

Hence, using equation (5), we have

$$\frac{\partial \phi}{\partial n} = -\phi, \quad 0 \leq x \leq \frac{\alpha}{2}(L + D) \text{ and } z = \infty. \quad (11)$$

A DRBEM is employed to obtain numerical solutions to equation (4) subject to boundary conditions (6) to (11). The DRBEM was initially proposed by Brebbia and Nardini [8]. Other researchers have been using this method to solve various boundary value problems, such as Zhu et al. [20], Ang [3], and Ang and Ang [4]. An integral equation for solving equation (4) is

$$\begin{aligned} \lambda(\xi, \eta) &= \int\int_R \varphi(x, z; \xi, \eta)\phi(x, z)\, dxdz \\ &+ \int_C \left[\phi(x, z)\frac{\partial}{\partial n}(\varphi(x, z; \xi, \eta)) - \varphi(x, z; \xi, \eta)\frac{\partial}{\partial n}(\phi(x, z))\right] ds(x, z) \quad (12) \end{aligned}$$

where

$$\varphi(x, y; \xi, \eta) = \frac{1}{4\pi} \ln[(x - \xi)^2 + (y - \eta)^2],$$

is the fundamental solution of the two-dimensional Laplace's equation, and lies on smooth part of $(C) \in R$

$$\lambda(\xi, \eta) = \begin{cases} \frac{1}{2}, & (\xi, \eta \\ 1, & (\xi, \eta \end{cases}$$

Integral equation (12) may be solved numerically. Boundary C is discretized by constant line elements, and a number of interior points are chosen as collocation points. The midpoints of every segments are chosen as collocation points besides the interior collocation points. Let N and M be the numbers of the line segments and the interior points respectively. $C^{(1)}$, $C^{(2)}$, $\cdots$, $C^{(N)}$ be the line segments, and

$(a^{(i)},b^{(i)})$ is the midpoint of $C^{(i)}$, $i = 1,2,\cdots,N$. Points $(a^{(N+1)},b^{(N+1)})$, $(a^{(N+2)},b^{(N+2)})$, $\cdots$, $(a^{(N+M)},b^{(N+M)})$ be the interior collocation points. The value of ϕ can be approximated by

$$\begin{aligned}\lambda(a^{(n)},b^{(n)})\phi^{(n)} &= \sum_{k=1}^{N+M} \nu^{(nk)}\phi^{(k)} \\ &+ \sum_{m=1}^{N}[\phi^{(m)}F_1^{(m)}(a^{(n)},b^{(n)}) - p^{(m)}F_2^{(m)}(a^{(n)},b^{(n)})], \\ &\quad n = 1,2,\cdots,N+M.\end{aligned}$$

Where

$$\begin{aligned}\nu^{(nk)} &= \sum_{i=1}^{N+M} \Upsilon(a^{(n)},b^{(n)};a^{(i)},b^{(i)})\omega^{(ik)}, \\ F_1^{(m)}(a^{(n)},b^{(n)}) &= \int_{C^{(m)}} \varphi(x,z;a^{(n)},b^{(n)})ds(x,z), \end{aligned}\tag{13}$$

and

$$F_2^{(m)}(a^{(n)},b^{(n)}) = \int_{C^{(m)}} \frac{\partial}{\partial n}(\varphi(x,z;a^{(n)},b^{(n)}))ds(x,z). \tag{14}$$

$\Upsilon(a^{(n)},b^{(n)};a^{(i)},b^{(i)})$ and $\omega^{(ik)}$ are defined as

$$\begin{aligned}\Upsilon(a^{(n)},b^{(n)};a^{(i)},b^{(i)}) &= \lambda(a^{(n)},b^{(n)})\chi(a^{(n)},b^{(n)};a^{(i)},b^{(i)}) \\ &+ \sum_{j=1}^{N} \frac{\partial}{\partial n}(\chi(x,z;a^{(i)},b^{(i)}))\Big|_{(x,z)=(a^{(j)},b^{(j)})} F_1^{(j)}(a^{(n)},b^{(n)}) \\ &+ \sum_{j=1}^{N} \chi(a^{(j)},b^{(j)};a^{(i)},b^{(i)})F_2^{(j)}(a^{(n)},b^{(n)}),\end{aligned}$$

and

$$[\omega^{(ik)}] = [\rho(a^{(k)},b^{(k)};a^{(i)},b^{(i)})]^{-1},$$

where

$$\begin{aligned}\rho(x,z;a^{(i)},b^{(i)}) &= 1 + (r(x,z;a^{(i)},b^{(i)}))^2 + (r(x,z;a^{(i)},b^{(i)}))^3 \\ \chi(x,z;a^{(i)},b^{(i)}) &= \frac{1}{4}(r(x,z;a^{(i)},b^{(i)}))^2 + \frac{1}{16}(r(x,z;a^{(i)},b^{(i)}))^4 \\ &\quad + \frac{1}{25}(r(x,z;a^{(i)},b^{(i)}))^2,\end{aligned}$$

and

$$r(x,z;a^{(i)},b^{(i)}) = \sqrt{(x-a^{(i)})^2 + (z-b^{(i)})^2}.$$

From the numerical values of ϕ obtained, The dimensionless MFP, Φ, is computed using equation (3).

4. Results and Discussion

The method described in the preceding section is tested on a problem involving steady infiltration from periodic flat channels in homogeneous pima clay loam (PCL). The value of both L and D are set to be 50 cm. The value of α for PCL is 0.014 cm^{-1} [2, 9].

One way to analyse the numerical solutions is to compare them with the analytic solution of the problem. For the values of L, D, and α set as mentioned above, the analytic solution is

$$\Phi(x,z) = \frac{5\pi}{7} + 2\sum_{n=1}^{\infty} \frac{l_n \sin(0.5n\pi)\cos(10n\pi x/7)}{nk_n}, \tag{15}$$

where

$$l_n = \exp\left\{ z\left[1 - \sqrt{1 + \left(\frac{10n\pi}{7}\right)^2}\right]\right\},$$

and

$$k_n = 0.35\left[1 + \sqrt{1 + \left(\frac{10n\pi}{7}\right)^2}\right].$$

The DRBEM is used to obtain numerical solutions of Φ, and implemented using MATLAB. In the implementation of the DRBEM, a numerical integration package in MATLAB, quadl, is used to compute the integrals in equations (13) and (14). In the DRBEM formulation, the domain must be bounded by a simple closed curve. Therefore, the value of z in the domain must satisfy $0 \le z \le c$, for a positive real number c. In this study, the value of c is set to be 4. In other words, $z = 4$ is an imposed boundary.

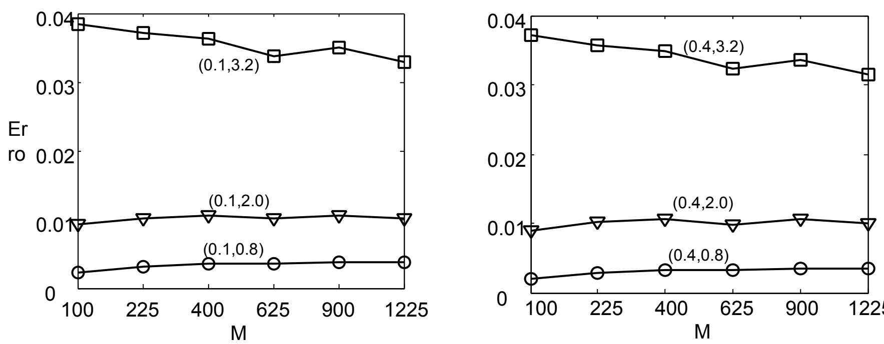
We first set the number of constant line elements, N, to be 200. Using this value of N, six different values of M are chosen. For easy reference, this set of M and N is called set A. We then fix the value of M, to be 625. With this value of M, different values of N are chosen, and this set of M and N is denoted by set B. Sets A and B are summarized in Table 1.

Table 1. Sets of constant line segments and interior collocation points.

Set *A*		Set *B*	
N	M	N	M
200	100	200	625
200	225	300	625
200	400	400	625
200	625	500	625
200	900	600	625
200	1225	700	625

Numerical solutions of Φ at six interior points (0.1,0.8), (0.4,0.8), (0.1,2.0), (0.4,2.0), (0.1,3.2) and (0.4,3.2) are obtained using the values of N and M in Table 1 above. These points are chosen, such that two points are at $z = 0.8$ (near the surface of soil), two are at $z = 2.0$ (middle of the domain in z-direction), and the two other are at $z = 3.2$ (near the imposed boundary). Using the analytic solution

(15), errors of the numerical solutions are computed, and shown graphically in Figures 2 and 3.

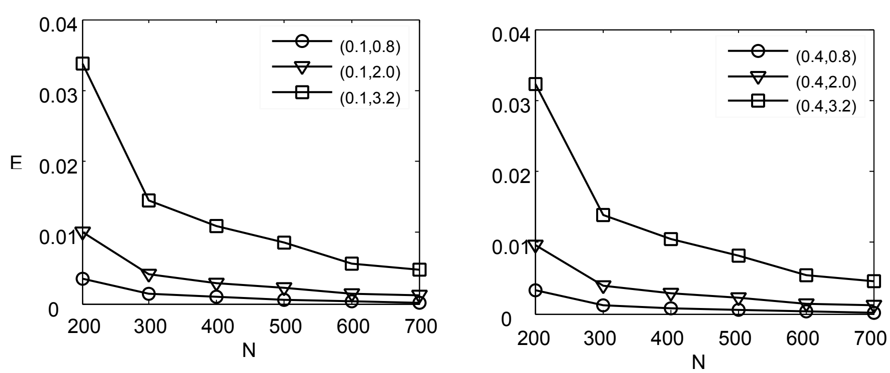


(a) Errors at three points along $x = 0.1$.(b) Errors at three points along $x = 0.4$.

Figure 2. Plot of dimensionless errors at selected points for $N = 200$ and six different values of M.

Figure 2 shows errors of Φ obtained at the six interior points, using N and M in Set A. It can be seen from Figure 2 that the errors increase as z increases. In particular, the errors at $z = 0.8$ are below 0.004, at $z = 2.0$ are about 0.01, and at $z = 3.2$ vary between 0.03 and 0.04. The larger errors could be due to the imposed boundary at $z = 4$.

It can also be seen that at $z = 0.8$ and $z = 2.0$, there are no significant differences in the errors obtained from the six different sets of N and M. In contrast, there are significant differences in the errors obtained at $z = 3.2$. The error decreases as M increases, and remains approximately constant when $M \geq 625$. These results indicate that a larger value of M tends to yield more accurate values of Φ than smaller values of M in the region beyond a certain depth. In addition, there is a threshold value of M, at which the accuracy may not be affected by any increase in the value of M. In this particular case, for $N = 200$, this threshold value for M is 625.



a) Errors at three points along $x = 0.1$. (b) Errors at three points along $x = 0.4$.

Figure 3. Plot of dimensionless errors at selected points for $M = 625$ and six different values of N.

Figure 3 shows errors of numerical solutions obtained using Set *B*. As before, the errors obtained increase as *z* increases. However, unlike the results in Figure 2, the increase in *N* affects the errors at the selected values of *z*. In particular, the errors decrease as *N* increases. The decrease in the error at some value of *z* is steeper than that at smaller values of *z*. This means that a value of *N* yields more accurate solutions than smaller values of *N*, especially when *z* approaches 4.

In the DRBEM construction, for *n* constant elements and *m* interior collocation points, the number of the line integrals (13) and (14) evaluated is $(n+m)\times 2n$. Thus, if for instance we have two pairs of *N* and *M*, which produce the same number of collocation points, the pair with larger value of *N* produces bigger number of such integrals.

As mentioned earlier, a built in numerical integration function quadl was used to compute the line integrals. It is found that quadl tends to take a considerable amount of time to compute these integrals. Hence, it is expected that a greater proportion of computational time is spent on evaluating the line integrals. Hence, a bigger number of such line integrals takes longer computational times than smaller one.

From the results presented, more accurate solutions are obtained by increasing the number of constant elements. Despite the accuracy of the solutions obtained, a number of constant line segments makes the computational time taken much longer than smaller numbers of constant line segments. In the study of infiltration from irrigation channels, researchers are normally interested in the solutions at shallow level of soil. A good accuracy at this level may be achieved by using small numbers of constant elements. Hence, small numbers of constant elements give more advantages in infiltration studies, especially in computational time to obtain solutions.

5. Concluding Remarks

A problem involving steady infiltration from periodic flat channels have been solved numerically using a DRBEM. Two sets of constant line elements, *M*, and interior collocation points, *N*, are considered to obtain numerical solutions. The first set contains six pairs of *M* and *N*, with fixed *M*, and the other set contains six other pairs with fixed *N*.

Using the solutions obtained, errors are computed using the analytic solution of the problem. From the errors obtained, the number of interior points may not affect the accuracy of the solutions at small values of *z*. However, when *z* approaches 4, the accuracy is affected by the number of interior collocation points. In contrast, the number of constant line elements gives impact on the accuracy, especially at *z* close to 4.

Using a higher number of constant elements tends to result in longer computational time. This is likely to be due to the fact that more line integrals need to be evaluated and compounded by the use of a built in function in MATLAB, quadl. Such problems may be overcome by using other mean of implementation, and more efficient schemes of numerical integration.

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References

[1] Denecke, K. and Wismath, S.L., *Universal Algebra and Coalgebra*, World Scientific, 2009.

[2] Amoozegar-Fard, A., Warric, A.W. and Lomen, D.O., ”Design nomographs for trickle irrigation system”, *J of Irr. and Drainage Eng.* **110** (1984), 107 - 120.

[3] Ang, W.T., ”A Laplace transformation dual-reciprocity boundary element method for a class of two-dimensional microscale thermal problems”, *Eng. Comput.* **19** (2002), 467 - 478.

[4] Ang, W.T. and Ang, K.C., ”A dual-reciprocity boundary element solution of a generalized nonlinear schrodinger equation”, *Published online 4 March 2004 in Wiley InterScience (www.interscience.wiley.com)*

[5] Azis, M.I., Clements, D.L., and Lobo, M., ”A boundary element method for steady infiltration from periodic channels”, *ANZIAM J.* **44(E)** (2003) C61 - C78.

[6] Basha, H.A. ”Multidimansional linearized nonsteady infiltration towar a shallow water table”, *Water Res. Research* **36** (2000) 2567 - 2573.

[7] Batu, V. ”Steady infiltration from single and periodic strip sources”, *Soil Sci. Soc. Am. J.* **42** (1978) 544 - 549.

[8] Brebbia, C.A. and Nardini, D., ”Dynamic analysis in solid mechanics by an alternative boundary element procedure”, *Int J Soil Dyn Earthquake Eng.* **2** (1983) 228 - 233.

[9] Bresler, E., ”Analysis of trickle irrigation with application to design problems”, *Irrigation Science* **1** (1978) 3 - 17.

[10] Clements, D.L., Lobo, M. and Widana, N., ”A hypersingular boundary integral equation for a class of problems concerning infiltration from periodic channels”, *Electronic Journal of Boundary Elem.*, **5** (2007) 1 - 16.

[11] Clements, D.L. and Lobo, M., ”A BEM for time-dependent infiltration from an irrigation channels”, *Eng Anal Boundary Elem.* **34** (2010) 1100 - 1104.

[12] Gardner, W.R., ”Some steady state solutions of the unsaturated moisture flow equation with application to evaporation from a water table”, *Soil Sci.* **85** (1957) 228 - 232.

[13] Lobo, M., Clements, D.L. and Widana, N., ”Infiltration from irrigation channels into soil with impermeable inclusion”, *ANZIAM J.* **46(E)** (2005) C1055 - C1068.

[14] Philip, J.R., ”Flow in porous media”, *Annu. Rev. Fluid Mech.* **2** (1970) 177 - 204.

[15] Philip, J.R., ”Steady infiltration from spherical cavities”, *Soil Sci. Soc. Am. J.* **48** (1984) 724 - 729.

[16] Pullan, A.J. and Collons, I.F., ”Two- and three-dimensional steady quasi-linear infiltration from buried and survace cavities using boundary element techniques”, *Water Res Research*, **23** (1987) 1633 - 1644.

[17] Pullan, A.J., ”Linearized time-dependent infiltration from a shallow pond”, *Water Resources Research.*, **28** (1992) 1041 - 1046.

[18] Weachter, R.T. and Philip, J.R., ”Steady two- and three-dimensional flows in unsaturated soil: the scattering analog”, *Water Resources Research*, **21**(1985) 1875 - 1887.

[19] Weachter, R.T. and Mandal, A.C., ”Steady infiltration from a semicircular cylindrical trench and a hemispherical pond into unsaturated soil”, *Water Resources Research*, **29** (1993) 457 - 467.

[20] Zhu, S.P., Satravaha, P. and Lu, X.P., ”Solving linear diffusion equations with the dual reciprocity method in Laplace space”, *Eng Anal Boundary Elem.*, **13** (1994) 1 - 10.

AN INTEGRATED INVENTORY MODEL WITH IMPERFECT-QUALITY ITEMS IN THE PRESENCE OF A SERVICE LEVEL CONSTRAINT

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Abstract. This article develops an integrated production-inventory model with imperfect quality items. The production process is imperfect and produces a certain number of defective items with a known probability density function. Our work is based on the paper of Lin [11]. We extend the model by considering a service level constraint and lead time and ordering cost reductions act dependently. It is assumed that shortages are allowed and partially backlogged on the buyer's side, and that the lead time demand is assumed to be normally distributed. The objective is to determine the optimal order quantity, reorder point, lead time and the number of deliveries simultaneously so that the expected total system cost is minimized. Furthermore, numerical example and sensitivity analysis are carried out to demonstrate the benefits of the model and to illustrate the effect of parameters on the decision and the total system cost.

Keywords and Phrases:Integrated model, imperfect quality, service level constraint, controllable lead time.

1. Introduction

The vendor-buyer integrated production inventory system received a lot of attention in recent years. Strategic vendor-buyer alliance is a formalized type of collaborative relationship between a vendor and a buyer in a supply chain. It involves commitment to long-term cooperation, shared benefits and cost, joint problem solving and information sharing. This close partnership will ultimately improve product quality and reduce inventory cost and lead time of the supply chain. Therefore, several authors (e.g., Ben-Daya and Hariga [2], Lin [12], Rad and Khoshalhan [16], Shah[a] and Shah[b] [17]) have presented the integrated inventory management system. Goyal [4] is among the first who analyzed an integrated inventory model for a single-vendor single-buyer system, in which he assumed that the vendor's production rate is infinite. Banerjee [1] modified Goyal's model and presented a joint economic-lot-size model where a vendor produces for a buyer to order on a lot-for-lot basis. Goyal [5] further generalized Banerjee's [1] model by relaxing the assumption of the lot-for-lot policy of the vendor and suggested that the vendor's economic production quantity should be a positive integer multiple of the buyer's purchase quantity.

The classical economic order quantity (EOQ) model assumes that items produced are of perfect quality, which is usually not the case in real production. In practice, it can often be observed that there are defective items being produced due to imperfect production process. Porteus [15] first incorporated the effect of defective items in the basic EOQ model. Recently, several authors (e.g., Hsu[1] and Hsu[2] [7], Hsu[a] and Hsu[b] [8],Lin [11]) consider imperfect quality item in integrated vendor-buyer inventory model. Lin [11] explored the lead time reductions and imperfect quality items problems on the integrated production-inventory system, where shortages are allowed with partial backorder. We notice that the possible relationship between ordering cost and lead time is ignored. In practices, the lead time and ordering cost reductions may be related closely; the reduction of lead time may accompany the reduction of ordering cost, and vice versa. For example, the implementation of electronic data interchange (EDI) can reduce both the lead time and ordering cost simultaneously (see, Chen et al. [3], Ouyang, et al. [14]). In integrated vendor-buyer models described above, the stock out cost is one of the components in the objective function. However, the stock out cost often includes intangible components such as loss of goodwill and potential delay to the other parts of the inventory system, so it is difficult to explicitly express the stock out cost. Instead of having a stock out cost term in the objective function, a service level constraint, which implies that the stock out level per cycle is bounded, is considered. Moreover, a service level criterion is generally easy to interpret and establish. Thus service level constraint models are more popular in real-life inventory systems than full-cost models (see, Lin [10], Ma and Qiu [13]).

Based on the survey above, in this paper, we extend Lin's [11] model by considering the service level constraint and the reduction of lead time accompanies a decrease of ordering cost. The objective is to determine the optimal order quantity, reorder point, lead time and the number of deliveries simultaneously so that the expected total system cost is minimized. By constructing Lagrange function, the analysis regarding the solution procedure is conducted, and a solution algorithm is then provided. Moreover, numerical examples and sensitivity analysis are carried out to demonstrate the benefits of the model and to illustrate the effects of parameters on the decision and the total system cost. This paper is organized as follows. In Section 2, the notation and assumptions used in this paper are introduced. In Section 3, we develop a mathematical model that integrates the vendor's and the buyer's expected total cost and takes into consideration imperfect-quality items and service level constraint. In Section 4, a Lagrange function and an efficient algorithm are constructed to find the optimal solutions. Section 5 provides a numerical example and discussion of the results. Finally, in Section 6 we draw some concluding and give suggestions for the future research.

2. Notations and Assumptions

In this paper, we propose an integrated vendor-buyer inventory model with imperfect-quality items and service level constraint, in which shortages are allowed with partial backorders. The model is developed using the following notations and assumptions.

2.1 Notation

D Expected demand per unit time on the buyer (for non-defective items).
P Production rate on the vendor.
F Transportation cost per order.
ω Unit treatment cost of defective items on the vendor.
π Fixed penalty cost per unit short on the buyer.
π_0 Marginal profit (i.e., cost of lost demand) per unit on the buyer.
L_0 Original length of lead time (before any crashing of lead time is made).
L The length of lead time (decision variable).
r Reorder point of the buyer for non-defective items (decision variable).
S Setup cost per set-up for the vendor.
h_{b1} Inventory holding cost per non-defective item per unit time for the buyer.
h_{b2} Inventory holding cost per defective item per unit time for the buyer $h_{b2} \leq h_{b1}$.
h_v Inventory holding cost per item per unit time for the vendor.
A_0 Original ordering cost (before any crashing of lead time is made).
A Ordering cost per order for the buyer.
α Proportion of demands which are not met from stock, that is, $1-\alpha$ is the service level.
β Fraction of the demand during the stock-out period that will be backordered,$0 < \beta < 1$.
x Screening rate on the buyer.
s Screening cost per unit for the buyer.
Q Order quantity of the buyer for non-defective items.
q Order quantity of the buyer per order including defective items, i.e., shipping quantityfrom the vendor to the buyer per shipment (decision variable).
n The number of lots in which the items are delivered from the vendor to the buyer in one production run, a positive integer (decision variable).
T_1 The period during which the vendor produces.
T_2 The period duringwhich the vendor supplies from inventory.
X The lead time demand with finite mean DL and standard deviation $\sigma\sqrt{L}$,
where σ denotes the standard deviation of the demand per unit time.
Y The number of defective items in a lot size$q$, a random variable.
$E[\cdot]$ Mathematical expectation.

2.2 Assumptions

1. There is single-buyer and single-vendor for a single-product in this model.
2. Inventory is continuously reviewed. The buyer places an order or requests for successive shipments when on hand inventory level (based on the number of non-defective items) falls to the reorder point$r$. The reorder point $r =$ expected demand during lead time + safety stock (SS), and $SS =$

k ×(standard deviation of lead time demand), i.e., $r = DL + k\sigma\sqrt{L}$, where k is known as the safety factor.

3. The lead time L has m mutually independent components. The i^{th} component has a minimum duration a_i and normal duration b_i, and a crashing cost per unit time c_i. The components can be rearranged such that $c_1 \le c_2 \le \cdots \le c_m$. The components are crashed starting from the least crashing cost per unit time.
4. Let L_i be the length of the lead time with component $1,2,\ldots,i$ crashed to their minimum duration and $L_i = \Sigma_{j=1}^{m} b_j - \Sigma_{j=1}^{i}\left(b_j - a_j\right), i = 1,2,\ldots,m$. Also, we let $L_0 = \Sigma_{j=1}^{m} b_j$ and $R(L) = c_i(L_{i-1} - L) + \Sigma_{j=1}^{i-1} c_j\left(b_j - a_i\right)$, the lead time crashing cost per cyclefor a given $L \in [L_i, L_{i-1}]$.
5. The extra costs incurred by the vendor will be fully transferred to the buyer if shortened lead time is requested.
6. The buyer orders a lot of size Q (for non-defective items) and will receive the batch quantity in n equally-sized shipments of size q, where n is a positive integer.
7. An arriving lot may contain some defective items. It is assumed that the number of defective items Y, in an arriving order of size q is a random variable which has a binomial distribution with parameters q and γ where γ $(0 \le \gamma < 1)$ represents the defective rate in the order lot. Upon the arrival of the order, all the items in the lot are inspected with the screening rate x by the buyer, and defective items in each lot are discovered and returned to the vendor at the time of delivery of the next lot.
8. Vendor's production rate for the non-defective items is greater than buyer's demand rate, i.e., $(1-\gamma)P > D$.
9. The screening process and the demand proceed simultaneously, but the screening rate is greater than the demand rate.

3. Mathematical Model

In this section, a mathematical model is formulated to minimize the joint total expected cost per unit time with service level constraint by finding the optimal order quantity, lead time and safety factor of the buyers and the number of shipments in a production cycle between the vendor and the buyer. The integrated inventory model is designed as follows. If the buyer orders quantity$q$, then the vendor producesnqat one set-up, with a finite production rate$P$, in order to reduce its set-up, where n is a positive integer. The production process of the vendor is assumed to deteriorate resulting in a random number of defective items, Y. Therefore, the expected length of each ordering cycle for the buyer is$E[T] = E[(q-Y)D^{-1}]$and the expected length of each production cycle for the vendor is$E[nT] = nE[(q-Y)D^{-1}]$. The buyer adopts 100% screening process to inspect all the items upon arrival and all the defective items in each lot are returned to the vendor at the time of delivery of the next lot.

3.1. Buyer's *expected average total cost per unit time*

Based on the above notations and assumptions, the inventory system of the buyer can be depicted in Figure 1. The first step in formulation of the total expected cost per unit time for the buyer is to derive the expressions of the separate cost components for the buyer. For the model without interaction between lead time and ordering cost and without service level constraint, we will closely follow the model in Lin [11]. Such that, holding cost for non-defective itemis $h_{b1}\left(\frac{q-y}{D}\right)\left\{\frac{Dqy}{2x(q-y)}+\frac{q-y}{2}+r-DL+(1-\beta)E[(X-r)^{+}]\right\}$, holding cost for defective items is $h_{b2}\left[\frac{(q-y)y}{D}-\frac{qy}{2x}\right]$, stock-out cost is $[\pi+\pi_0(1-\beta)]E[(X-r)^{+}]$where$E[(X-r)^{+}]$is the expected number of shortages at the end of the cycle, screening cost is sq, and lead time crashing cost is $R(L)$. The buyer's total cost per cycle, given that there are$y$defective items in an arriving shipment of size$q$, is the sum of the ordering cost, transportation cost, holding cost for non-defective item, holding cost for defective items, stock-out cost, screening cost, and lead time crashing cost. Symbolically, the buyer's total cost per cycle can be represented by:

$$C_b(q,r,L;y)$$
$$=A+F+h_{b1}\left(\frac{q-y}{D}\right)\left\{\frac{Dqy}{2x(q-y)}+\frac{q-y}{2}+r-DL+(1-\beta)E[(X-r)^{+}]\right\}$$
$$+h_{b2}\left[\frac{(q-y)y}{D}-\frac{qy}{2x}\right]+[\pi+\pi_0(1-\beta)]E[(X-r)^{+}]+sq+R(L), \qquad (1)$$

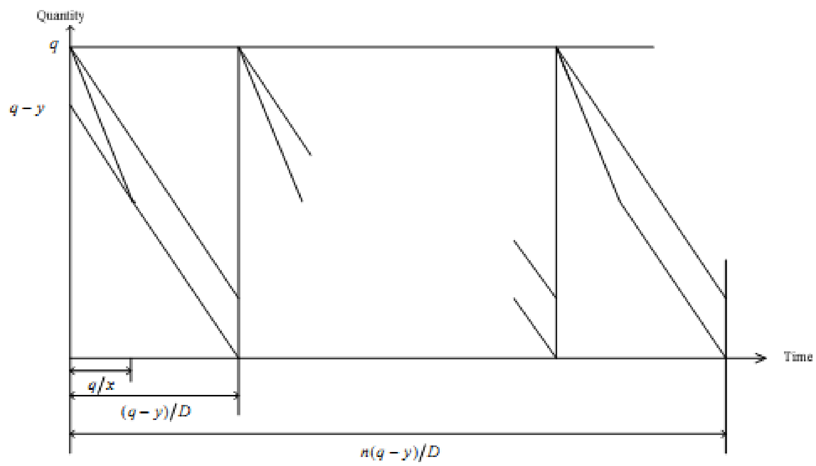

Figure 1. The inventory system for the buyer

The number of defective items in a lot$q$follows a binomial random variable with parameters$q$andγ, where$\gamma(0\le\gamma<1)$represents the defective rate in an order lot. That is,

$$P_r(Y)=C_Y^q\gamma^Y(1-\gamma)^{q-Y}, \text{ for } Y=0,1,2,\dots,q.$$

Therefore, one has

$$E[Y]=q\gamma \text{ and } E[Y^2]=q^2\gamma^2+q\gamma(1-\gamma).$$

The expected length of the cycle time is$E[T]=E\left[\frac{q-Y}{D}\right]=\frac{q(1-\gamma)}{D}$and therefore the expected average total cost per unit time for the buyer is

$$EC_b^U(q,r,L)=\frac{E[C_b(q,r,L;Y)]}{E[T]}$$
$$=\frac{D}{q(1-\gamma)}\{A+F+\pi+\pi_0(1-\beta)E[(X-r)^{+}]+R(L)\}$$
$$+\frac{h_{b1}q\gamma D}{2x(1-\gamma)}+\frac{h_{b1}(q-q\gamma+\gamma)}{2}+h_{b1}\{r-DL+(1-\beta)E[(X-r)^{+}]\}$$
$$+h_{b2}\gamma(q-1)-\frac{h_{b2}q\gamma D}{2x(1-\gamma)}+\frac{sD}{1-\gamma}. \qquad (2)$$

According to the opinion of Chen et al. [3] and Ouyang et al. [14], the reduction of lead time may accompany the reduction of ordering cost, and vice versa. In this subsection, we consider that the lead time and ordering cost reductions have the following logarithmic functional relationship:

$$\frac{A_0-A}{A_0} = \tau \ln\left(\frac{L}{L_0}\right), \tag{3}$$

where $\tau(< 0)$ is constant scaling parameter for the logarithmic relationship between percentages of reductions in lead time and ordering cost. In this case, the ordering cost A can be written as

$$A(L) = d + e\ln(L), \tag{4}$$

where $d = A_0 + \tau A_0 \ln(L_0)$ and $e = -\tau A_0 > 0$. Substituting (4) into (2), the model (2) can be rewritten as

$$EC_b^U(q,r,L) = \frac{D}{q(1-\gamma)}\{d + e\ln(L) + F + \pi + \pi_0(1-\beta)E[(X-r)^+] + R(L)\}$$
$$+\frac{h_{b1}q\gamma D}{2x(1-\gamma)} + \frac{h_{b1}(q-q\gamma+\gamma)}{2} + h_{b1}\{r - DL + (1-\beta)E[(X-r)^+]\}$$
$$+h_{b2}\gamma(q-1) - \frac{h_{b2}q\gamma D}{2x(1-\gamma)} + \frac{sD}{1-\gamma}. \tag{5}$$

For a given safety factor which satisfies the probability that lead time demand at the buyer exceed the reorder point, the actual proportion of demands not met from stock should not exceed the desired value of α. Therefore, the service level constraint can be established as

$$\frac{\text{Expected demand shortages at the end of cycle}}{\text{Expected quantity avaliable for satisfying the demand per cycle}} \leq \alpha.$$

Symbolically, the service level constraint can be formulated as $\frac{E[(X-r)^+]}{E[q-Y]} = \frac{E[(X-r)^+]}{q(1-\gamma)} \leq \alpha$. Using this definition and replacing the stock out cost by a definition on the service level in (5), the problem is transformed to

$$\text{Minimize}\, EC_b^U(q,r,L) = \frac{D}{q(1-\gamma)}\{d + e\ln(L) + F + R(L)\} + \frac{h_{b1}q\gamma D}{2x(1-\gamma)} + \frac{h_{b1}(q-q\gamma+\gamma)}{2}$$
$$+h_{b1}\{r - DL + (1-\beta)E[(X-r)^+]\} + h_{b2}\gamma(q-1)$$
$$-\frac{h_{b2}q\gamma D}{2x(1-\gamma)} + \frac{sD}{1-\gamma}, \tag{6}$$

subject to $\frac{E[(X-r)^+]}{q(1-\gamma)} \leq \alpha$.

4.2. Vendor's expected average total cost per unit time

The production rate of vendor's non-defective items is greater than the buyer's demand rate and then the vendor's inventory level will increase gradually. When the total required amount nq is fulfilled, the vendor stops producing immediately. Therefore, the vendor's inventory per production cycle can be obtained by subtracting the accumulated buyer inventory level from the accumulated vendor inventory level. Figure 2 shows the vendor's holding cost per cycle can be obtained as (see, for example, Hsu[a] and Hsu[b] [7] and Lin [12]):

Holding cost per cycle

$= h_v$ [bold area – shaded area]

$$= h_v\left\{\left[nq\left(\frac{q}{P} + (n-1)T\right) - \frac{nq\left(\frac{nq}{P}\right)}{2}\right] - T[q + 2q + \cdots + (n-1)q]\right\}$$
$$= h_v\left\{\frac{nq}{2}\left\{\left[(n-1)T + \frac{q}{P}\right] + \left[(n-1)T + \frac{q}{P} - \frac{nq}{P}\right]\right\} - [q + 2q + \cdots + (n-1)]qT\right\}$$
$$= h_v\left\{\frac{nq^2}{P} + \frac{n(n-1)qT}{2} - \frac{n^2q^2}{2P}\right\}. \tag{7}$$

The total cost of the vendor per production cycle includes the set-up cost, defective item treatment cost,and holding cost. One has

$$C_v(q,n) = S + nY\omega + h_v\left[\frac{nq^2}{P} + \frac{n(n-1)qT}{2} - \frac{n^2q^2}{2P}\right]. \quad (8)$$

Therefore, the expected average total cost per unit time for the vendor can be obtained as

$$\begin{aligned} EC_v^U(q,n) &= \frac{E[C_v(q,n)]}{E[nT]} \\ &= \left\{S + nq\gamma\omega + h_v\left[\frac{nq^2}{P} + \frac{n(n-1)q^2(1-\gamma)}{2D} - \frac{(nq)^2}{2P}\right]\right\} \times \frac{D}{nq(1-\gamma)} \\ &= \frac{SD}{nq(1-\gamma)} + \frac{D\gamma\omega}{1-\gamma} + \frac{h_v Dq}{1-\gamma}\left[\frac{1}{P} + \frac{(n-1)(1-\gamma)}{2D} - \frac{n}{2P}\right]. \quad (9)\end{aligned}$$

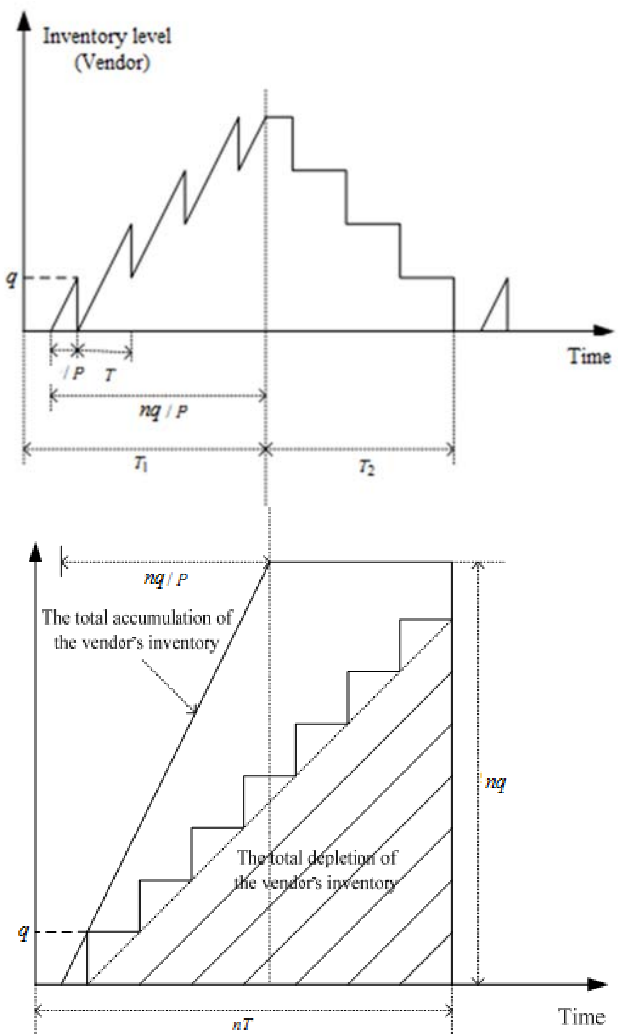

Figure 2. Time-weighted inventory for the vendor

4.3. *The joint total expected average cost per unit time*

The joint expected average total cost per unit time consists of the expected total cost of the buyer and the expected total cost of the vendor. Therefore, we have the following problem:

$$\text{Minimize} JEC^U(q,r,L,n) = EC_b^U(q,r,L) + EC_v^U(q,n)$$
$$= \frac{D}{q(1-\gamma)}\left[\frac{S}{n} + A + F + R(L)\right]$$
$$+\frac{D}{1-\gamma}\left[s + \gamma\omega - \frac{h_{b2}q\gamma}{2x} + h_v q\left(\frac{1}{P} + \frac{(n-1)(1-\gamma)}{2D} - \frac{n}{2P}\right)\right]$$
$$+h_{b1}\left[\frac{q\gamma D}{2x(1-\gamma)} + \frac{(q - q\gamma + \gamma)}{2} + r - DL + (1-\beta)E[(X-r)^+]\right]$$
$$+h_{b2}\gamma(q-1), \tag{10}$$

subject to $\frac{E[(X-r)^+]}{q(1-\gamma)} \le \alpha$.

Note that when the lead time and ordering cost reductions are assumed to act independently and there is no service level constraint, model (10) can be reduced to Lin's [11] model.

As mentioned earlier, it is assumed that the lead time demand X follows a normal distribution with finite mean DL and standard deviation $\sigma\sqrt{L}$. We note that $r = DL + \sigma\sqrt{L}$, and hence, the expected shortage quantity, $E[(X-r)^+]$, where $x^+ = \max\{x, 0\}$, can be written as

$$E[(X-r)^+] = \int_k^\infty \sigma\sqrt{L}(z-k)\phi(z)dz = \sigma\sqrt{L}\Psi(k) > 0, \tag{11}$$

where $\Psi(k) = \phi(k) - k[1-\Phi(k)]$. Notations ϕ and Φ denote the standard normal probability density function and cumulative distribution function, respectively. Substituting (11) into (10) and using the safety factor k as a decision variable instead of r, the model (10) can be transformed to

$$\text{Minimize} JEC^U(q,k,L,n) = \frac{D}{q(1-\gamma)}\left[\frac{S}{n} + d + e\ln(L) + F + R(L)\right]$$
$$+\frac{D}{1-\gamma}\left[s + \gamma\omega - \frac{h_{b2}q\gamma}{2x} + h_v q\left(\frac{1}{P} + \frac{(n-1)(1-\gamma)}{2D} - \frac{n}{2P}\right)\right]$$
$$+h_{b1}\left[\frac{q\gamma D}{2x(1-\gamma)} + \frac{(q - q\gamma + \gamma)}{2} + k\sigma\sqrt{L} + (1-\beta)\sigma\sqrt{L}\Psi(k)\right]$$
$$+h_{b2}\gamma(q-1), \tag{12}$$

subject to $\frac{\sigma\sqrt{L}\Psi(k)}{q(1-\gamma)} \le \alpha$.

4. Optimal Solution

The problems are to find the optimal values of $q, k, L,$ and n such that $JEC^U(q,k,L,n)$ in (12) is minimum and satisfying the service level constraint. The inequality constraint in (12) can be converted into equality by adding a slack variable $H^2 \ge 0$. One has

$$\text{Minimize} JEC^U(q,k,L,n) = \frac{D}{q(1-\gamma)}\left[\frac{S}{n} + d + e\ln(L) + F + R(L)\right]$$
$$+\frac{D}{1-\gamma}\left[s + \gamma\omega - \frac{h_{b2}q\gamma}{2x} + h_v q\left(\frac{1}{P} + \frac{(n-1)(1-\gamma)}{2D} - \frac{n}{2P}\right)\right]$$

$$+h_{b1}\left[\frac{q\gamma D}{2x(1-\gamma)}+\frac{(q-q\gamma+\gamma)}{2}+k\sigma\sqrt{L}+(1-\beta)\sigma\sqrt{L}\Psi(k)\right]$$
$$+h_{b2}\gamma(q-1), \tag{13}$$

subject to $\sigma\sqrt{L}\Psi(k)+H^2=q(1-\gamma)\alpha$.

The Lagrange function of (13) is

$$JEC^U(q,k,L,n,\lambda,H)=\frac{D}{q(1-\gamma)}\left[\frac{S}{n}+d+e\ln(L)+F+R(L)\right]$$
$$+\frac{D}{1-\gamma}\left[s+\gamma\omega-\frac{h_{b2}q\gamma}{2x}+h_v q\left(\frac{1}{P}+\frac{(n-1)(1-\gamma)}{2D}-\frac{n}{2P}\right)\right]$$
$$+h_{b1}\left[\frac{q\gamma D}{2x(1-\gamma)}+\frac{(q-q\gamma+\gamma)}{2}+k\sigma\sqrt{L}+(1-\beta)\sigma\sqrt{L}\Psi(k)\right]$$
$$+h_{b2}\gamma(q-1)+\lambda\left[\sigma\sqrt{L}\Psi(k)+H^2-q(1-\gamma)\alpha\right], \tag{14}$$

where λ is the Lagrange multiplier.

For any given (q,k,n,λ,H), $JEC^U(q,k,L,n,\lambda,H)$ is a concave down function in variable L for $L\in[L_i,L_{i-1}]$, because

$$\frac{\partial^2 JEC^U(q,k,L,n,\lambda,H)}{\partial L^2}=-\frac{De}{q(1-\gamma)L^2}-\frac{1}{4L\sqrt{L}}h_{b1}\sigma\left[k+(1-\beta)\Psi(k)+\frac{\lambda}{h_{b1}}\Psi(k)\right]<0. \tag{15}$$

Hence, for fixed (q,k,n,λ,H) the minimum expected joint total cost will occur at the boundary points $L=L_i$ and $L=L_{i-1}$.

Taking the first partial derivatives of (14) with respect to q,k,λ, and H respectively and equalizing the results to zero, we have

$$\frac{\partial JEC^U(q,k,L,n,\lambda,H)}{\partial q}=-\frac{D}{q^2(1-\gamma)}\left[\frac{S}{n}+d+e\ln(L)+F+R(L)\right]$$
$$+\frac{D}{1-\gamma}\left[-\frac{h_{b2}\gamma}{2x}+h_v\left(\frac{1}{P}+\frac{(n-1)(1-\gamma)}{2D}-\frac{n}{2P}\right)\right]$$
$$+h_{b1}\left[\frac{\gamma D}{2x(1-\gamma)}+\frac{1-\gamma}{2}\right]+h_{b2}\gamma-\lambda(1-\gamma)\alpha=0, \tag{16}$$

$$\frac{\partial JEC^U(q,k,L,n,\lambda,H)}{\partial k}=h_{b1}\sigma\sqrt{L}\left[1-[1-\Phi(k)]\left(1-\beta+\frac{\lambda}{h_{b1}}\right)\right]=0, \tag{17}$$

$$\frac{\partial JEC^U(q,k,L,n,\lambda,H)}{\partial \lambda}=\sigma\sqrt{L}\Psi(k)+H^2-q(1-\gamma)\alpha=0, \tag{18}$$

and

$$\frac{\partial JEC^U(q,k,L,n,\lambda,H)}{\partial H}=2H\lambda=0. \tag{19}$$

From (19), if $H>0$, then λ ought to be equal to zero. Substituting $\lambda=0$ into (17), it is reduce to

$$h_{b1}\sigma\sqrt{L}[1-[1-\Phi(k)](1-\beta)]=0. \tag{20}$$

An unreasonable solution, $\Phi(k)=-\frac{\beta}{1-\beta}<0$ (since the value of a CDF cannot be negative) is obtained when (20) is solved for k. Hence $H=0$ and $\lambda\neq 0$ can be expected and the constraint $\frac{\sigma\sqrt{L}\Psi(k)}{q(1-\gamma)}\leq\alpha$ is active when the optimal solution is obtained.

Solving equations (16) to (18) respectively, we have the following results:

$$q=\sqrt{\frac{2D\left(\frac{S}{n}+d+e\ln[L]+F+R(L)\right)}{h_v\left((n-1)(1-\gamma)+\frac{(2-n)D}{P}\right)+h_{b1}(1-\gamma)^2+\frac{D\gamma(h_{b1}-h_{b2})}{x}+2h_{b2}\gamma(1-\gamma)-2\alpha(1-\gamma)^2\lambda}}, \tag{21}$$

$$\lambda^*=\frac{h_{b1}[\beta+(1-\beta)\Phi(k)]}{1-\Phi[k]}, \tag{22}$$

and

$$\Psi[k^*]=\frac{\alpha q[1-\gamma]}{\sigma\sqrt{L}}. \tag{23}$$

Substituting (22) into (21) yields

$$q^* = \sqrt{\frac{2D\left(\frac{S}{n}+d+e\ln[L]+F+R(L)\right)}{h_v\left((n-1)(1-\gamma)+\frac{(2-n)D}{P}\right)+h_{b1}(1-\gamma)^2+\frac{D\gamma(h_{b1}-h_{b2})}{x}+2h_{b2}\gamma(1-\gamma)-2\alpha(1-\gamma)^2\left(\frac{h_{b1}[\beta+(1-\beta)\Phi(k)]}{1-\Phi[k]}\right)}}. \tag{24}$$

The following proposition shows that, for fixed n and $L \in [L_i, L_{i-1}]$, the point (q^*, k^*) is the local optimal solution, which satisfies the active constraint $\frac{\sigma\sqrt{L}\Psi(k)}{q(1-\gamma)} = \alpha$ and minimizes the expected average total cost $JEC^U(q, k, L, n)$.

Proposition 1. *For given n and $L \in [L_i, L_{i-1}]$, the point (q^*, k^*) satisfies the second order sufficient condition (SOSC) for the minimizing problem with a single constraint.*

PROOF. For fixed n and $L \in [L_i, L_{i-1}]$, the constraint $\frac{\sigma\sqrt{L}\Psi(k)}{q(1-\gamma)} \le \alpha$ is active when the optimal solution is obtained (i.e. slack variable $H = 0$). Therefore, in what follows, we show that (q^*, k^*) satisfies the second order sufficient condition for the minimizing problem with a single equality constraint. We first obtain the bordered Hessian matrix **H** as follows:

$$\mathbf{H} = \begin{bmatrix} 0 & \frac{\partial^2 JEC^U(q,k,L,n,\lambda,H)}{\partial\lambda\partial q} & \frac{\partial^2 JEC^U(q,k,L,n,\lambda,H)}{\partial\lambda\partial k} \\ \frac{\partial^2 JEC^U(q,k,L,n,\lambda,H)}{\partial q\partial\lambda} & \frac{\partial^2 JEC^U(q,k,L,n,\lambda,H)}{\partial q^2} & \frac{\partial^2 JEC^U(q,k,L,n,\lambda,H)}{\partial q\partial k} \\ \frac{\partial^2 JEC^U(q,k,L,n,\lambda,H)}{\partial k\partial\lambda} & \frac{\partial^2 JEC^U(q,k,L,n,\lambda,H)}{\partial k\partial q} & \frac{\partial^2 JEC^U(q,k,L,n,\lambda,H)}{\partial k^2} \end{bmatrix}.$$

where

$$\frac{\partial^2 JEC^U(q,k,L,n,\lambda,H)}{\partial q^2} = \frac{2D}{q^3(1-\gamma)}\left[\frac{S}{n} + d + e\ln(L) + F + R(L)\right],$$

$$\frac{\partial^2 JEC^U(q,k,L,n,\lambda,H)}{\partial q\partial k} = \frac{\partial^2 JEC^U(q,k,L,n,\lambda,H)}{\partial k\partial q} = 0,$$

$$\frac{\partial^2 JEC^U(q,k,L,n,\lambda,H)}{\partial q\partial\lambda} = \frac{\partial^2 JEC^U(q,k,L,n,\lambda,H)}{\partial\lambda\partial q} = -(1-\gamma)\alpha,$$

$$\frac{\partial^2 JEC^U(q,k,L,n,\lambda,H)}{\partial k^2} = h_{b1}\sigma\sqrt{L}\left(1-\beta+\frac{\lambda}{h_{b1}}\right)\phi(k),$$

$$\frac{\partial^2 JEC^U(q,k,L,n,\lambda,H)}{\partial k\partial\lambda} = \frac{\partial^2 JEC^U(q,k,L,n,\lambda,H)}{\partial\lambda\partial k} = -h\sigma\sqrt{L}[1-\Phi(k)].$$

For a given value of n and L, since there are two variables (q, k) and one constraint, therefore, we need to check the sign of the last one principal minor determinant of **H** at point (q^*, k^*). If the sign of it is negative, then this point satisfies the second order sufficient condition for the minimizing problem (see, Taha [19]). Now we proceed by checking the sign of the last one principal minor determinant of **H** at point (q^*, k^*).

$$|H_{33}| = |H|$$
$$= -\left(h\sigma\sqrt{L}[1-\Phi(k)]\right)^2 \frac{2D}{q^3(1-\gamma)}\left[\frac{S}{n} + d + e\ln(L) + F + R(L)\right]$$
$$-h_{b1}\sigma\sqrt{L}\left(1-\beta+\frac{\lambda}{h_{b1}}\right)\phi(k)\left((1-\gamma)\alpha\right)^2 < 0.$$

Since the sign of $|H_{33}|$ is negative, hence, it can be concluded that the optimal value (q^*, k^*) satisfies the sufficient condition for the minimizing problem with a constraint.□

Note that there is a negative term in the denominator of q^*. Recall that we assume $\gamma < 1 - D/P$, that is, $1-\gamma > D/P$. We have

$$h_v\left((n-1)(1-\gamma)+\frac{(2-n)D}{P}\right) > h_v\left(\frac{(n-1)D}{P}+\frac{(2-n)D}{P}\right) = h_v\frac{D}{P} > 0.$$

Thus, in order to have a feasible q solution (the denominator of q^* is positive), the maximum allowable value of λ in (30) is

$$\lambda < \frac{h_v\left((n-1)(1-\gamma)+\frac{(2-n)D}{P}\right)+h_{b1}(1-\gamma)^2+\frac{D\gamma(h_{b1}-h_{b2})}{x}+2h_{b2}\gamma(1-\gamma)}{2\alpha(1-\gamma)^2}. \tag{25}$$

The problem is to determine the value of n that minimizes $JEC^U(q,k,L,n)$. If we take the second derivatives of (14) with respect to n, we have

$$\frac{\partial^2 JEC^U(q,k,L,n,\lambda,H)}{\partial n^2} = \frac{2Ds}{q(1-\gamma)n^3} > 0. \tag{26}$$

Therefore, $JEC^U(q,k,L,n,\lambda,H)$ is a convex function of n, for fixed q,k,L,λ, and H. Because the number of shipments per batch production run, n, is a discrete variable, the optimal value of n (denoted by n^*) can be obtained when

$$JEC^U(q^*,k^*,L^*,n^*) \le JEC^U(q^*,k^*,L^*,n^*-1),$$

and

$$JEC^U(q^*,k^*,L^*,n^*) \le JEC^U(q^*,k^*,L^*,n^*+1).$$

Once we obtain the n^* value, the optimal order quantity of the buyer for non-defective items is given by $Q^* = n^*E[q^*-Y] = n^*q^*(1-\gamma)$.

We note that an explicit general solution for (q^*,k^*) is not easy to obtain because the evaluation for equations (23) and (24) requires knowledge of the value of the other. The optimal value (q^*,k^*) can be obtained by using a similar graphical technique used in Hadley and Whitin [6]. The same numerical search technique also has been used in Lin [11], Lin [12], and others. Therefore, we now develop an algorithm to find the optimal values for lot size, reorder point, lead time, and the number of shipments per production run from the vendor to the buyer.

Algorithm.

Step 1. Set $n=1$.

Step 2. For each L_i, $i=0,1,2,\dots,m$, perform (i) to (v).

- (i) Start with $k_{i1}=0$ and get $\Phi(k_{i1})=0.5$.
- (ii) Substituting $\Phi(k_{i1})$ into (24) to evaluate q_{i1}.
- (iii) Using q_{i1} determine $\Psi(k_{i2})$ from (23).
- (iv) Check $\Psi(k_{i2})$ from Silver and Peterson [18] to find k_{i2}, and then $\Phi(k_{i2})$.
- (v) Repeat (ii) to (iv) until no changes occurs in the values of q_i and k_i.
- (vi) Compute the corresponding $JEC^U(q_i,k_i,L_i,n)$, $i=0,1,2,\dots,m$.

Step 3. Find $\min\{JEC^U(q_i,k_i,L_i,n), i=1,2,\dots,m\}$.

If $JEC^U(q_n^*,k_n^*,L_n^*,n) = \min\{JEC^U(q_i,k_i,L_i,n), i=1,2,\dots,m\}$, then the point (q_n^*,k_n^*,L_n^*,n) is the optimal solution for fixed n.

Step 4. Set $n=n+1$, and repeat Step 2 to Step 3 to get $JEC^U(q_n^*,k_n^*,L_n^*,n)$.

Step 5. If $JEC^U(q_n^*,k_n^*,L_n^*,n) \le JEC^U(q_{n-1}^*,k_{n-1}^*,L_{n-1}^*,n-1)$, then go to Step 4, otherwise go to Step 6.

Step 6. Set $JEC^U(q_n^*,k_n^*,L_n^*,n) = JEC^U(q_{n-1}^*,k_{n-1}^*,L_{n-1}^*,n-1)$. Then (q^*,k^*,L^*,n^*) is the optimal solution, the optimal reorder point is $r^* = DL^* + k^*\sigma\sqrt{L^*}$ and the optimal effective order quantity (i.e., the optimal quantity of non-defective items) is $Q^* = n^* q^* (1-\gamma)$.

If the vendor and the buyer do not work together in a cooperative manner towards minimizing their mutual total costs, both the vendor and the buyer determine their inventory policy independently. The buyer use model (6) to compute his economic order quantity, reorder point and lead time. Substituting (11) and $r = DL + k\sigma\sqrt{L}$ into (6), model (6) can be transformed to

$$\text{Minimize } EC_b^U(q,k,L) = \frac{D}{q(1-\gamma)}\{d + e\ln(L) + F + R(L)\} + \frac{h_{b1} q\gamma D}{2x(1-\gamma)} + \frac{h_{b1}(q - q\gamma + \gamma)}{2}$$
$$+ h_{b1}\{k\sigma\sqrt{L} + (1-\beta)\sigma\sqrt{L}\Psi(k)\} + h_{b2}\gamma(q-1)$$
$$- \frac{h_{b2} q\gamma D}{2x(1-\gamma)} + \frac{sD}{1-\gamma}, \tag{27}$$

subject to $\frac{\sigma\sqrt{L}\Psi(k)}{q(1-\gamma)} \le \alpha$.

Applying the above solution processes developed in this section, it can be shown that the minimum expected average total cost for buyer will occur at the end points of the interval $[L_i, L_{i-1}]$, and for fixed $L \in [L_i, L_{i-1}]$, the buyer's independent optimal solution is

$$\lambda^* = \frac{h_{b1}[\beta + (1-\beta)\Phi(k)]}{1 - \Phi[k]} \tag{28}$$

$$\Psi[k^*] = \frac{\alpha q[1-\gamma]}{\sigma\sqrt{L}} \tag{29}$$

$$q^* = \sqrt{\frac{2D\big(d + e\ln[L] + F + R(L)\big)}{h_{b1}(1-\gamma)^2 + \frac{D\gamma(h_{b1} - h_{b2})}{x} + 2h_{b2}\gamma(1-\gamma) - 2\alpha(1-\gamma)^2\left(\frac{h_{b1}[\beta + (1-\beta)\Phi(k)]}{1-\Phi[k]}\right)}}. \tag{30}$$

For a given buyer's optimal order quantity, the vendor will determine the optimal number of deliveries, n, so that his expected average total cost in (8) is minimum. Since the number of deliveries is a positive integer, we compute equation (8) by setting $n = 1,2,3,\ldots$. The optimal value of n for vendor and $EC_v^U(q^*, n^*)$ can be obtained.

5. Numerical Example

In this section, a numerical example is utilized to demonstrate the feasibility of the proposed solution procedure. Consider an integrated vendor-buyer cooperative inventory model with the following data: $D = 600$ units/year, $P = 2000$ units/year, $A = \$200$/order, $S = \$1500$ /setup, $h_v = \$2$/unit/year, $h_{b1} = \$4$ /unit/year, $h_{b2} = \$3$ /unit/year, $F = \$25$/shipment, $\omega = \$4$/unit, $s = \$0.5$/unit, $x = 175200$ unit/year, $\sigma = 7$ units/week, $\beta = 0.8$, $\gamma = 0.025$. We assume that the maximum allowable proportion of demands which are not met from stock is $\alpha = 0.015$ (i.e. the worst service level is $1 - \alpha = 0.985$). It is assumed that the lead time demand is normally distributed and the lead time is divided into three components as shown in Table 1.

Table 1. Lead time data

Lead time Component i	Normal duration b_i (days)	Minimum duration a_i (days)	Unit fixed crashing cost c_i (\$ per day)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

The constant scaling parameter τ of the logarithmic relationship between lead time and ordering cost reductions has five different values. There are $0, -0.2, -0.5, -0.8$ and -1 respectively. Algorithm procedure is applied to yield the results for various τ values as shown in Table 2. The optimal policy for each τ value can be determined by comparing $JEC^U(q_n^*, k_n^*, L_n^*, n)$, $n = 1,2,\ldots$, and the results are summarized in Table 3. Moreover, we also list the results of fixed ordering cost model (i.e., take $\tau = 0$) in the same table in order to observe the relationships between lead time and ordering cost. From the results in Table 3, as the value of τ decreases, the integrated policy has the higher frequency of deliveries, smaller lot size, smaller ordering cost, and lower expected average total cost (the larger savings of the total cost).

If the inventory policies of the vendor and the buyer are determined independently, in the case of $\tau = -0.5$, we obtain $q^* = 244.74$ units, $k^* = 0.23$ (i.e. $r^* = 40$ units) and $L^* = 3$ weeks for the buyer and $n^* = 5$ for the vendor. The expected average total cost for the buyer is $EC_b^U(q^*, k^*, L^*) = \$1171.62$ and the expected average total cost for the vendor is $EC_v^U(q^*, n^*) = \$1568.92$. The joint total cost of the non-integrated model is \$2740.44. For the same τ value, the expected average total cost in an integrated model is \$2205.98. It is lower than the joint total cost in non-integrated model.

Table2.The results for various τ using the solution procedures (L_n^* in weeks)

τ	n	L_n^*	$R(L_n^*)$	$A(L_n^*)$	q_n^*	k_n^*	r_n^*	$JEC^U(q_n^*, k_n^*, L_n^*, n)$
0	1	3	57.4	200.00	649.687	−0.62	27	2258.45
	2	3	57.4	200.00	432.924	−0.22	32	2258.68
−0.2	1	3	57.4	160.77	642.273	−0.61	27	2237.62
	2	3	57.4	160.77	424.409	−0.21	32	2221.70
	3	3	57.4	160.77	331.214	0.00	35	2289.76
−0.5	1	3	57.4	101.92	630.890	−0.59	29	2232.32
	2	3	57.4	101.92	411.020	−0.18	34	2205.98
	3	3	57.4	101.92	317.284	0.03	36	2262.34
−0.8	1	3	57.4	43.070	619.317	−0.57	28	2200.95
	2	3	57.4	43.070	397.235	−0.15	33	2149.83
	3	3	57.4	43.070	302.603	0.07	36	2182.87
−1.0	1	3	57.4	3.8300	611.568	−0.56	28	2179.36
	2	3	57.4	3.8300	387.807	−0.13	33	2111.27
	3	3	57.4	3.8300	292.401	0.10	36	2127.99

Table 3. Summary of the optimal integrated policy for various τ (L^* in weeks)

τ	n^*	L^*	$A(L^*)$	q^*	k^*	r^*	$JEC^U(q^*, k^*, L^*, n^*)$	Saving(%)
0	1	3	200.00	649.687	−0.62	27	2258.45	−
−0.2	2	3	160.77	424.409	0.00	32	2221.70	1.63
−0.5	2	3	101.92	411.020	−0.18	34	2205.98	2.32
−0.8	2	3	43.070	302.603	0.07	36	2149.83	4.81
−1.0	2	3	3.8300	387.807	−0.13	33	2111.27	6.52

Note: Saving is based on the fixed ordering cost model (i.e., $\tau = 0$).

A fair and acceptable profit sharing mechanism is the key to the success of an integrated model. A profit sharing mechanism is first suggested by Goyal [4]. The total annual cost $JEC^U(q_n^*, k_n^*, L_n^*, n^*)$ should be allocated to the vendor and the buyer as follows:

$$\rho = \frac{EC_b^U(q^*,k^*,L^*)}{EC_b^U(q^*,k^*,L^*)+EC_v^U(q^*,n^*)},$$

cost to the buyer $= \rho JEC^U(q^*,k^*,L^*,n^*)$,

cost to the vendor $= (1-\rho)JEC^U(q^*,k^*,L^*,n^*)$.

For example, when$\tau = -0.5$, $\rho = \frac{1171.62}{1171.62+1568.92} = 0.4275$. The allocated buyer's total cost is\$943.09 and the allocated vendor's total cost is\$1262.89. The non-integrated policy and integrated policy are shown in Table 4. The allocations for various values of τ are summarized in Table 5. With profit sharing, it is shown that the integrated policy reduces the joint total cost.

Table 4. Non-integrated policy and the total cost shaving for the case of $\tau = -0.5$

Non-integrated model		Integrated model	
Buyer's order quantity	244.74	Buyer's order quantity	411.020
Lead time	3	Lead time	3
Safety factor (reorder point)	40	Safety factor (reorder point)	34
Vendor's production quantity	2000	Vendor's production quantity	2000
Vendor's setup cost	1500	Vendor's setup cost	1500
Buyer's total cost	1171.62	Buyer's total cost	1412.04
		Allocated buyer's total cost	943.09
Vendor's total cost	1568.92	Vendor's total cost	1595.47
		Allocated vendor's total cost	1262.89
Joint cost	2740.44	Joint cost	2205.98

Table 5. Allocation of the total cost for each case of τ

τ	Non-integrated model			Integrated model				
				Buyer		Vendor		JEC^U
	EC_b^U	EC_v^U	JEC^U	EC_b^U	Allocated total cost	EC_v^U	Allocated total cost	
0	1409.8	1557.4	2967.2	1775.4	1073.0	1682.2	1185.3	2258.4
−0.2	1323.3	1553.3	2876.7	1429.1	1022.0	1573.4	1199.6	2221.7
−0.5	1171.6	1568.9	2740.4	1412.0	943.00	1595.4	1262.8	2205.9
−0.8	987.30	1584.5	2571.9	1395.0	825.30	1620.6	1324.5	2149.8
−1.0	831.40	1595.7	2427.2	1383.7	723.20	1639.4	1388.0	2111.2

Next, the value of lower bound of service level (1- α) is varied from 0.96 to 0.99 with equal interval 0.1 to perform sensitivity analysis and give some observations and managerial implications. From the results in Table 6, it is seen that the increasing 1- α value results in smaller lot size, higher safety factor which implies higher safety stock and reorder point, and higher expected average annual total cost. It seems that a lower service level benefits manager in profit. However, the lower service will produce negative influences on brand and customer loyalty which are crucial to building competitive advantage in market. From this perspective, the manager should determine a proper lower bound of service level which can balance short-term income and long-term development.

Table 6. Effect of change in parameter α

1- α	n^*	L^*	q^*	k^*	r^*	$JEC^U(q^*, k^*, L^*, n^*)$
0.96	1	3	662.820	-2.13	9	2170.37
0.97	1	3	661.082	-1.57	16	2195.94
0.98	1	3	655.856	-0.97	23	2230.82
0.99	1	3	637.278	-0.21	32	2303.34

6. Concluding Remarks

This paper extends the Lin's [11] model by considering that the reduction of lead time accompanies a decrease of ordering cost. Additionally, the stock-out cost component in the objective function is replaced by a condition on a service level to formulate the model. The production process is imperfect and produces a random number of defective items in buyer's arrival order lot. We consider the policy in which the delivery quantity to the buyer is identical at each shipment. Once the buyer receives the lot, a 100% screening process of the lot is conducted with a fixed screening rate. It is assumed that the lead time demand and the number of defective item in an order lot follow the normal distribution and binomial distribution, respectively. A Lagrange function and an efficient algorithmic approach are proposed to find the optimal order quantity, safety factor, lead time of the buyer and number of lot delivered from the vendor to the buyer in a production cycle while minimizing the joint total expected cost of the vendor-buyer integrated system and satisfying the service level constraint on the buyer. Moreover, the results contained in this research are illustrated and verified by a numerical example. In the future research, we may consider a capital investment in the reduction of setup cost.Another feasible extension of the present research is to follow Khan et al. [9] by considering the possibility of incorrectly classifying non-defective item as defective (a type I error), or incorrectly classifying a defective item as defective (a type II error). Thus, some defective items may be sold to customers, who in turn will detect the quality problem and return them to the buyer.

References

[1] Banerjee, A., 1986, A Joint Economic-Lot-Size Model for Purchaser and Vendor, *Decision Sciences*, 17, 292-311.

[2] Ben-Daya, M. and Hariga, M., 2004, Integrated Single Vendor Single Buyer Model with Stochastic Demand and Variable Lead Time, *International Journal of Production Economics*, 92, 75-80.

[3] Chen, C-K., Chang, H-C. and Ouyang, L-Y., 2001, A Continuous Review Inventory Model with Ordering Cost Dependent on Lead Time, *Information and Management Sciences,* Volume 12, Number 3, pp. 1-13

[4] Goyal, S. K., 1976, An Integrated Inventory Model for a Single Supplier-Single Customer Problem, *International Journal of Production Research,* 15(1), 107-111.

[5] Goyal, S. K., 1988, A Joint Economic-Lot-Size Model for Purchaser and Vendor: A Comment, *Decision Sciences*, 19, 236-241.

[6] Hadley, G. and Whitin, T., 1963, *Analysis of Inventory Systems*, New Jersey: Prentice Hall.

[7] Hsu[1], J-T. and Hsu[2], L-F., 2012, An Integrated Vendor-Buyer Cooperative Inventory Model for Items with Imperfect Quality and Shortage Backordering, *Advances in Decision Sciences*, Volume 2012m Article ID 679083.

[8] Hsu[a], J-T. and Hsu[b], L-F., 2013, An Integrated Vendor-Buyer Inventory Model with Imperfect Items and Planned Back Orders, *International Journal Adv Manuf Technol,* DOI 10.1007/s00170-013-4810-7.

[9] Khan, M., 2011, An Economic Order Quantity (EOQ) for Items with Imperfect Quality and Inspection Errors, *International Journal of Production Economics,* vol. 133, no.1, pp. 113-118.

[10] Lin, H-J., 2012, Effective Investment to Reduce Setup Cost in a Mixture Inventory Model Involving Controllable Backorder rate and Variable Lead Time with a Service Level Constraint, *Mathematical Probles in Engineering,* Volume 2012, Article ID 689061, doi: 10.1155/2012/689061.

[11] Lin, H-J., 2013, An Integrated Supply Chain Inventory Model with Imperfect-Quality Items, Controllable Lead Time and Distribution-Free Demand, *Yugoslav Journal of Operations Research,* 23 (2013), Number 1, 87 - 109.

[12] Lin, Y-J., 2009, An Integrated Vendor-Buyer Inventory Model with Backorder Price Discount and Effective Investment to Reduce Ordering Cost, *Computer and Industrial Engineering,* 56, 1597-1606.

[13] Ma, W-M. and Qiu, B-B., 2012, Distribution-Free Continuous Review Inventory Model with Controllable Lead Time and Setup Cost in the Presence of a Service Level Constraint, *Mathematical Problems in Engineering*, Volume 2012, Article ID 867847.

[14] Ouyang L-Y., Chuang, B-R., Lin, Y-J., 2007, The Inter-Dependent Reductions of Lead Time and Ordering Cost in Periodic Review Inventory Model with Backorder Price Discount, *Information and Management Sciences,* Volume 18, Number 3, pp. 195-208.

[15] Porteus, E., 1986, Optimal Lot Sizing, Process Quality Improvement and Setup Cost Reduction, *Operational Research*, 34, 137-144.

[16] Rad, M. A. and Khoshalhan, F., 2011, An Integrated Production-Inventory Model with Backorder and Lot for Lot Policy, *International Journal of Industrial Engineering and Production Research*, Volume 22, Number 2, pp. 127-134.

[17] Shah[a], N. H., and Shah[b], D. B., 2013, Joint Optimal Policy for Variable Deteriorating Inventory System of Vendor-Buyer Ordering with Trapezoidal Demand, *British Journal of Applied Science and Technology*, 3(1), 160-173.

[18] Silver, E.A. and Petersen, R., 1985,*Decision Systems for Inventory Management and Production Planning*, New York, John Wiley.

[19] Taha, H. M., 1997, *Operations Research An Introduction*, Prentice Hall, New Jersey.

HYBRID MODEL OF IRRIGATION CANAL AND ITS CONTROLLER USING MODEL PREDICTIVE CONTROL

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Abstract. In this paper, we formulate a hybrid model of irrigation canal that contains four reaches where each of reach has two state events. These state events of this hybrid model are triggered by the height of the water level which has two different surfaces. We formulate this hybrid model in piecewise affine (PWA) form, transform it into mixed logical dynamic (MLD) form using hybrid system description language (HYSDEL) that was embedded in hybrid toolbox for MATLAB and control this MLD using model predictive control (MPC). We control this irrigation canal so that the water level on each reach will be located at the desired level. Finally, we simulate this system and its controller to desire given desired level or set point. From the simulation results, the water level of all reaches are located at the desired level.

Key words and Phrases: Hybrid system, piecewise-affine (PWA) system, mixed logical dynamic (MLD) system, Model predictive control, irrigation canal

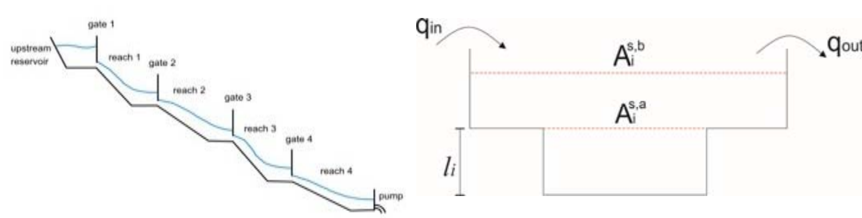
1. INTRODUCTION

Following [4], hybrid system is a mathematical model that consists of dynamics of real valued variables, discrete variables and their interaction. A particular case of discrete hybrid systems is piecewise-affine (PWA) systems whose mode depends on the current location (or event) of the state vector. Hybrid system in PWA form can be transformed into equivalent mixed logical dynamical (MLD) form [2] and it can be done using HYSDEL [14] that was embedded in hybrid control toolbox for MATLAB [2]. This MLD form is more suitable for solving optimization problem corresponding to the optimal control problem [4]. The state of an MLD model contains state and input of the PWA model and some auxiliary variables i.e. binary variable and real variable. Furthermore, besides the original constraint of PWA, MLD has auxiliary constraint in the form of matrix inequality. The controllability and observability of this hybrid model was given by [3] in order to be controllable and observable by a control design.

Hybrid predictive control is a famous control method that was applied in some problems like optimal semi-active suspension [5], direct injection stratified charge engines [6] and the stability of MPC for hybrid system was guaranteed by [7]. Predictive control can be applied to control an MLD system with some modifications such as the corresponding optimization problem. Predictive control for MLD system for finite time horizon can be done by forming the vector prediction of all state variables of MLD over time horizon, substituting them into objective function and solving the corresponding optimization problem. In case where the objective function has quadratic form, the corresponding optimization can be solved using mixed integer quadratic programming (MIQP) [2].

Irrigation canal is needed to be controlled to ensure that the water level of each reach is located at the desired level. To control this system, we need the mathematical model of the system. For flat irrigation canal (there is no different surface for any water level), the mathematical model was given by [13] in the linear discrete dynamics (non-hybrid model) and the control designs for this system were given by some researches like coordinated DMPC [11], distributed model predictive control [10, 12] and its performance analysis was given by[8]. For non-flat irrigation canal (there is different surfaces for some water levels), we need a new mathematical model formulation, that is a new hybrid model in order to control this system.

In this paper, we formulate the hybrid model of irrigation canal that have two different surfaces for each reach. This hybrid model will be presented as PWA form and it will be transformed equivalently into MLD form in order to be more suitable for designing its controller. We will apply MPC to control this MLD model. MPC for MLD contains a mixed integer quadratic optimization and we will solve it using MIQP. To observe how this hybrid model and its controller are working, we simulate this system so that the water level of each reach will be located at the desired level and show the output that is the water level of each reach.



(a) Irrigation canal with their surface areas four reaches
(b) A reach with two state events triggered by

Figure 1. Irrigation canal and a reach

2. MODELING OF HYBRID SYSTEM OF IRRIGATION CANAL

A hybrid model presents the dynamic of a system that constituted by parts described by logical parameter such as on/off switches, event states, if-then-else rules, etc [4]. Firstly, we will formulate the hybrid model of irrigation canal in the PWA form. Consider an irrigation canal system illustrated by Figure (1a) that has four reaches. Let $x_i(k)$ be the water level in meter unit (m) for reach-i, A_i^s be the water surface area (m^2) for reach-i, q_i^{in}-and q_i^{out} be the in-flow (m^3/s) and out-flow (m^3/s) for reach-i which are measured on upstream and downstream respectively and p_4 be the flow that passing the pump on reach-4. Let T_s be the sampling time, then the dynamic for reach-i can be written as [13].

$$x_i(k+1) = x_i(k) + \frac{T_s}{A_i^s} q_i^{in}(k) - \frac{T_s}{A_i^s} q_i^{out}(k) \quad (1)$$

For reach-i, let $A_i^{s,b}$ be the water surface area if $x_i > l_i$ (mode-0) and $A_i^{s,a}$ be the water surface area if $x_i \le l_i$ (mode-1) as illustrated by Figure (1b). Then the dynamic of reach i can be written as the following PWA model

$$x_i(k+1) = \begin{cases} x_i(k) + \frac{T_s}{A_i^{s,b}} q_i^{in}(k) - \frac{T_s}{A_i^{s,b}} q_i^{out}(k) & \text{if } x_i > l_i, \text{(mode-0)} \\ x_i(k) + \frac{T_s}{A_i^{s,a}} q_i^{in}(k) - \frac{T_s}{A_i^{s,a}} q_i^{out}(k) & \text{if } x_i \le l_i, \text{(mode-1)} \end{cases}$$

$$y_i(k) = x_i(k),$$

for $i = 1,2,3$ and especially for $i = 4$, q_4^{out} is replaced by p_4.

3. MODEL PREDICTIVE CONTROL FOR MLD SYSTEM

Model predictive control is one of a modern control design that has been applied by many researches to solve an optimal control problem including optimal control problem for a hybrid system. Let s be the number of the model parts or events and k denote the time step, then PWA model for discrete time in general form can be described as follows [4].

$$x_c(k+1) = \begin{cases} A_1 x_c(k) + B_1 u_c(k) & \text{if } \delta_1 = 1, \\ A_2 x_c(k) + B_2 u_c(k) & \text{if } \delta_2 = 1, \\ \vdots \\ A_s x_c(k) + B_s u_c(k) & \text{if } \delta_s = 1. \end{cases}$$

$$y_c(k) = Cx_c(k) + Du_c(k) \quad (2)$$

where $x_c(k) \in \mathbb{R}^{nc}$, $u_c(k) \in \mathbb{R}^{mc}$ and $y(k) \in \mathbb{R}^{pc}$ denote the state, input and output vectors respectively at time step k, $i = 1,2,...,s$, $\delta_i \in \{0,1\}$, and matrices A_i, B_i, C and D are real matrices with appropriate dimensions. This PWA model can be transformed into equivalent MLD and vice versa as follows [1]. Let $\delta \in \{0,1\}$ be the binary variable, $n_c > 0$ be the number of the state of the system then the MLD model can be described as follows.

$$x(k+1) = Ax(k) + B_1u(k) + B_2\delta(k) + B_3z(k) \quad (3)$$

$$y(k) = Cx(k) + D_1u(k) + D_2\delta(k) + D_3z(k) \quad (4)$$

$$E_2\delta(k) + E_3z(k) \le E_1u(k) + E_4x(k) + E_5 \quad (5)$$

where $x(k) = \begin{bmatrix} x_c(k) \\ x_l(k) \end{bmatrix}$ is the new state vector for MLD model, $x_c(k) \in \mathbb{R}^{n_c}$, $x_l(k) \in \{0,1\}^{n_l}$, $y(k) = \begin{bmatrix} y_c(k) \\ y_l(k) \end{bmatrix} \in \mathbb{R}^{p_c} \times \{0,1\}^{p_l}$ is the new output vector for MLD model, $u(k) = \begin{bmatrix} u_c(k) \\ u_l(k) \end{bmatrix} \in \mathbb{R}^{m_c} \times \{0,1\}^{m_l}$ is the new input vector for MLD model, $z(k) \in \mathbb{R}^{r_c}$ and $\delta(k) \in \{0,1\}^{r_l}$ are auxiliary variables. A, B_i, C, D_i and E_i are real constant matrices and E_5 is a real vector.

To control this MLD model, that is determining some input values such that the output track some desired set point or reference trajectories, we will use MPC controller by assuming that this MLD model is controllable and reachable. The objectives of the MPC can be described as the state gain to the set point or references. MPC minimizes this objective function that can be described as the following optimization [4].

$$\min_{[u,\delta,z]^T} J = \sum_{t=0} \|u(k) - u_r\|^2_{Q_2} + \|\delta(k) - \delta_r\|^2_{Q_3} + \|z(k) - z_r\|^2_{Q_4} \tag{6}$$

$$+ \|x(k) - x_r\|^2_{Q_1} + \|y(k) - y_r\|^2_{Q_5}\Big] \tag{7}$$

subject to :

$$x(k+1) = Ax(k) + B_1u(k) + B_2\delta(k) + B_3z(k) \tag{8}$$

$$-E_4x(k) - E_1u(k) + E_2\delta(k) + E_3z(k) \le E_5 \tag{9}$$

where T be the prediction horizon length, Q_1, Q_2, Q_3, Q_4, and Q_5 are the symmetric and positive definite weighting matrices, x_r, u_r, δ_r, and z_r are the references and $\|v\|^2_Q = v^TQv$.

The optimization (6) can be transformed into mixed integer quadratic (MIQ) optimization by forming the vector predictions of state, input u, δ, and z over horizon prediction T and substituting them into the objective function. This transformation gives the following MIQ optimization as follows.

$$\mathbf{R}^{*} = \arg\min_{\mathbf{R}} \mathbf{R}^0 S_1 \mathbf{R} + 2(S_2 + x^0_0 S_3)\mathbf{R} \tag{11}$$

subject to :

$$F_1\mathbf{R} \le F_2 + F_3x_0 \tag{12}$$

$$\mathbf{ABR} = x_f - A^Tx_0, \tag{13}$$

$$\mathbf{R} = [u(0),\ldots,u(T-1),\ldots,\delta(0),\ldots, \tag{14}$$

$$\delta(T-1), z(0),\ldots,z(T-1)]^T \tag{15}$$

where S_1, S_2, and S_3 are the real constant matrices with appropriate dimension, $\mathbf{A} = [A^{T-1}, A^{T-2}, \ldots, A^2, A, I]$, $\mathbf{B} = [\mathbf{B_1}\ \mathbf{B_2}\ \mathbf{B_3}]$, $F_1 = \mathbf{E} - \mathbf{E_4}G_4$, $F_2 = \mathbf{E_5}$, $F_3 = \mathbf{E_4}G_0$, $\mathbf{E} = [-\mathbf{E_1}\ \mathbf{E_2}\ \mathbf{E_3}]$, $G_0 = [I, A, A^2, \ldots, A^{T-1}]'$, $E_5 = [E_5, E_5, \ldots, E_5]'$,

E_i ... B_i

E

$E_i =$ ... $i = 1,2,3,4$, and $\mathbf{B}_i =$ ... $i = 1,2,3.$

B_i

E_i

Optimization (11) can be solved using MIQP that embedded in hybrid toolbox [2]. The optimal value $\mathbf{R}^{\square} = [u^{\square},\delta^{\square},z^{\square}]$ obtained from (11) contains the optimal value for $u^{\square}$, $\delta^{\square}$, and $z^{\square}$ over time horizon. The control action that will be applied to the system is $u^{\square}$ at current time step.

4. SIMULATION RESULTS

We simulate the hybrid model of irrigation canal that contains four reaches associates to (3)-(4) by applying MPC controller that the optimal input values are obtained by optimization (11) with two initial state values. For all i = 1,2,3,4, the parameter values for this system are $l_i = 2$, $A^{s,a}{}_i = 600$, $A^{s,b}{}_i = 800$, and $0 \leq u_i(k) \leq 4$.
The first simulation is using initial state or initial water level $x(0) = [1.3,2,2.5,1.8]^T$. The simulation results including desired water levels (set points) for all reaches are appeared on Figure (3-4).
Figure (2) shows the optimal control actions for all reaches generated by MPC. These control actions was determined by MIQP programming (11). At time step k(second), the value $u_1(k)$ presents the inflow (m^3/s) for reach-1 that passed through the gate-1 and analogously for $u_i(k)$ for i = 2,3,4,5. These control actions are applied to the system and give the outputs (water level for all reaches) that given by Figure (3).

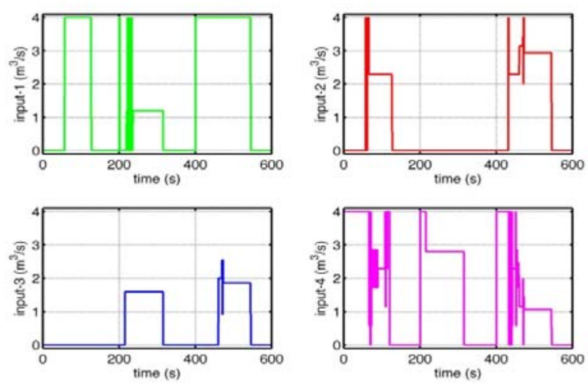


Figure 2. Optimal control inputs where $x(0) = [1.3,2,2.5,1.7]$

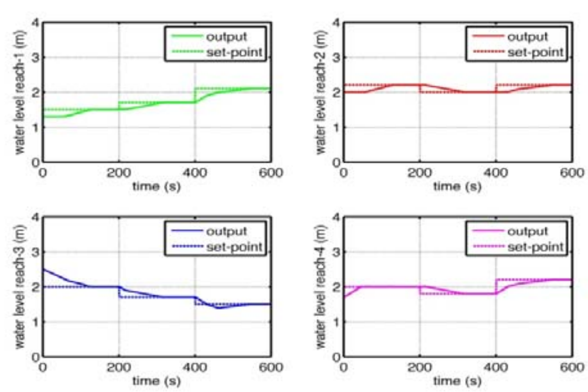


Figure 3. Water levels for all reaches where $x(0) = [1.3,2,2.5,1.7]$

Figure (3) shows the water level for all reaches. For reach-1, the initial water level is 1.3 m. From this figure, it can be seen that the water level is located at the desired level after 100 s and analogously for reach-i for $i = 2,3,4$. From this figure, it can be conclude that the controller brings the water levels of all reaches to the desired levels very well.

Figure (4) shows the modes for all reaches along simulation. For reach-1, the initial water level is 1.3 m, lower than $l_1 = 2m$, it means that this state is located at mode-1. At time step 600 s, the water level of reach-1 is 2.1 m, upper than

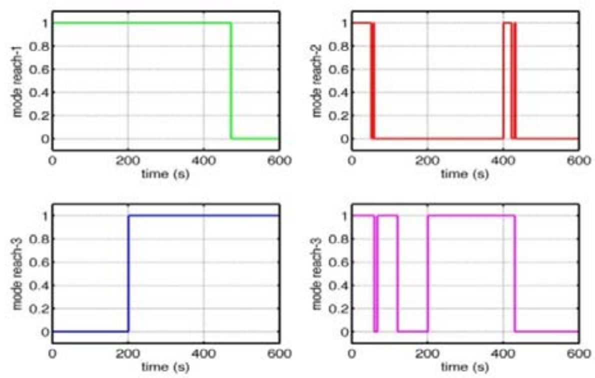


Figure 4. Modes for all reaches where $x(0) = [1.3, 2, 2.5, 1.7]$

$l_1 = 2m$, it means that this level is located at mode-0. We can observe analogously for other reaches. These mode values are dependent to the water level showed by Figure (3).

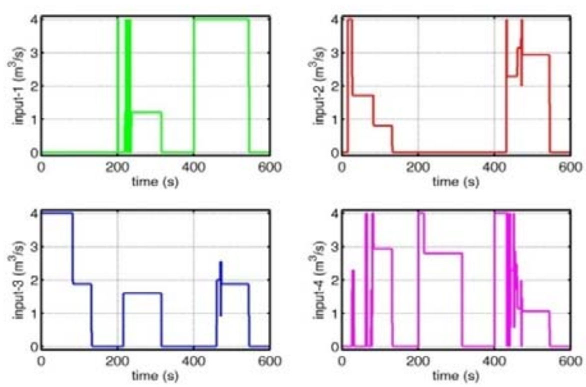


Figure 5. Optimal control inputs where $x(0) = [1.8, 2.5, 1.6, 2.2]$

The second simulation is using initial state $x(0) = [1.8, 2.5, 1.6, 2.2]$. Analogously to the first simulation, Figure (5) shows the optimal input values for all reaches, Figure (6) shows the water level for all reaches and Figure (7) shows the modes for all reaches. From Figure (6), it can be observed that the output of this system i.e. the water level for each reach followed the given desired level or set point. The mode for each reach on Figure (7) is following the output values on

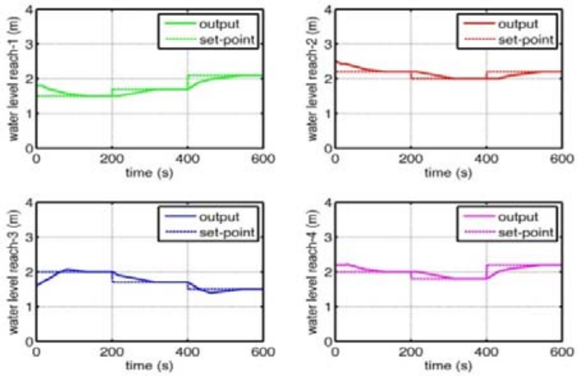


Figure 6. Water levels for all reaches where $x(0) = [1.8, 2.5, 1.6, 2.2]$

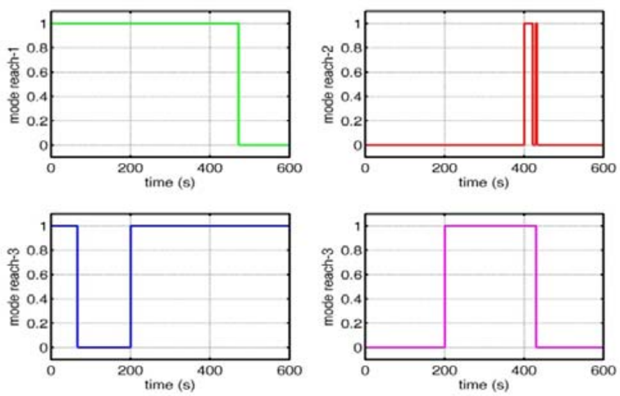


Figure 7. Modes for all reaches where $x(0) = [1.8, 2.5, 1.6, 2.2]$

Figure (6), that is if the water level is lower or equal to 2 m, then the mode is 1 and if the water level is upper than 2 m, then the mode is 0. These mode values are dependent to the water level showed by Figure (6).

5. CONCLUSIONS

In this work, the modeling and control of irrigation canal with two state events were considered. The mathematical model of this system was presented as hybrid model in the PWA form and it can be transformed equivalently into MLD form using HYSDEL and *mld* function on hybrid toolbox for MATLAB. The controller for this MLD model was designed using MPC for MLD. From the simulation results, the water level for all reaches were followed the given desired levels very well.

REFERENCES

[1] Bemporad, A., 2004, Efficient Conversion of Mixed Logical Dynamical Systems Into an Equivalent Piecewise Affine Form, *IEEE Transactions On Automatic Control*, Vol. 49, No. 5, May 2004., pp. 832838.

[2] Bemporad, A., *Hybrid Toolbox - Users Guide*, April 20, 2012.

[3] Bemporad, A., Ferrari-Trecate, G., and Morari, M., ”Observability and Controllability of Piecewise Affine and Hybrid Systems”, *IEEE Transactions on Automatic Control*, vol. 45, no. 10, October 2000.

[4] Borrelli, F., Bemporad, A., Morari, M., 2011, *Predictive Control for linear and hybrid systems*, June 16, 2011.

[5] Giorgetti,N. , Bemporad,A., Tseng,H.E. , and Hrovat,D. , Hybrid model predictive control application towards optimal semi-active suspension, *Int. J. Control*, vol. 79, no. 5, pp. 521533, 2006.

[6] Giorgetti, N., Ripaccioli,G., Bemporad,A., Kolmanovsky, I.V., and Hrovat,D., Hybrid model predictive control of direct injection stratified charge engines, *IEEE/ASME Transactions on Mechatronics*, vol. 11, no. 5, pp. 499506, Aug. 2006.

[7] Lazar, M., Heemels,W. P. M. H., Weiland, S., and Bemporad, A.., ”Stabilizing Model Predictive Control of Hybrid Systems”, *IEEE Transactions on Automatic Control*, vol. 51, no. 11, November 2006.

[8] Li, Y., and De Schutter, B., Performance analysis of irrigation channels with distributed control, *Proceedings of the 2010 IEEE International Conference on Control Applications, Yokohama, Japan*, pp. 21482153, Sept. 2010.

[9] Maciejowski, J.M., 2002, *Predictive Control with Constraints*, Prentice Hall Inc., England.

[10] Negenborn, R.R., and De Schutter, B., A distributed model predictive control approach for the control of irrigation canals, *Proceedings of the International Conference on Infrastructure Systems 2008: Building Networks for a Brighter Future, Rotterdam, The Netherlands*, 6 pp., Nov. 2008. Paper 152.

[11] Negenborn, R.R., van Overloop, P.J., and De Schutter, B., Coordinated distributed model predictive reach control of irrigation canals, *Proceedings of the European Control Conference 2009, Budapest, Hungary*, pp. 14201425, Aug. 2009.

[12] Negenborn, R.R., van Overloop, P.-J., Keviczky, T., and De Schutter, B., Distributed model predictive control of irrigation canals, *Networks and Heterogeneous Media*, vol. 4, no. 2, pp. 359380, June 2009.

[13] Overloop, V., 2006, *Model Predictive Control on Open Water Systems*, Ph.D Thesis, Delft University Press, The Netherlands.

[14] Torrisi, F.D., and Bemporad, A., HYSDEL A tool for generating computational hybrid models, *IEEE Trans. Contr. Systems Technology*, vol. 12, no. 2, pp. 235249, Mar. 2004.

FUZZY EOQ MODEL WITHTRAPEZOIDAL AND TRIANGULAR FUNCTIONS USING PARTIAL BACKORDER

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Abstract. EOQ fuzzy model is EOQ model that can estimate the cost from existing information. Using trapezoid fuzzy functions can estimate the costs of existing and trapezoid membership functions has some points that have a value of membership . $T\tilde{R}C$ value results of trapezoid fuzzy will be higher than usual TRC value results of EOQ model . This paper aims to determine the optimal amount of inventory in the company, namely optimal Q and optimal V, using the model of partial backorder will be known optimal Q and V for the optimal number of units each time a message . EOQ model effect on inventory very closely by using EOQ fuzzy model with triangular and trapezoid membership functions with partial backorder. Optimal Q and optimal V values for the optimal fuzzy models will have an increase due to the use of trapezoid and triangular membership functions that have a different value depending on the requirements of each membership function value. Therefore, by using a fuzzy model can solve the company's problems in estimating the costs for the next term.

Key words and Phrases: Inventory, EOQ, Fuzzy Logic, Partial Backorder, Membership function.

1. Introduction

Inventory control can be defined as activities and measures that are used to determine the right amount to meet the supply of an item. For the effective management of inventory greatly affect a company's profits. Common problems in the inventory system is an important issue for a company. Therefore in order to control inventory can be resolved properly used the "economic order quantity". Economic order quantity is the volume or number of purchases made on the most economical for each purchase. With the EOQ method , the company will reduce the inventory that has been improperly used or damaged and minimize the constraints of a production company to produce, or work delays and missed opportunities in the company sells its products due to run out of stock, then it must be determined from the value of EOQ (economic order quantity) to minimize the cost of the annual inventory. The basis of the EOQ formula introduced by Haris 1993. **[2]**

$$EOQ = \sqrt{\frac{2RC}{H}}$$

where: R = annual demand, C = order cost, H = holding cost per year.

In the EOQ model there are two EOQ model that is deterministic inventory model is characterized by the demand characteristics and arrival time before orders can be known with certainty, it is a basic assumption for deterministic models and probabilistic models are characterized by the demand and arrival time orders can not be previously known with certainty. However, in the real case many things to consider and many EOQ modeling studies using probabilistic approach to deal with uncertainty. Assumptions for the probabilistic model is characterized by demand and lead time that can not be known in advance with certainty so that needs to be approached with a probability distribution. But a lot of inventory parameter uncertainty and lack of cost information, to solve this problem it will be used fuzzy model because it can estimate the cost of the existing lack of cost information that already exists.

Fuzzy logic is a logic that has a value of vagueness or ambiguity between right and wrong, the fuzzy logic a value can be true and false at the same time, but how much truth and error a value depends on the weight of its membership. Membership function is a curve that shows the mapping of points of data input into its membership, which has a value between 0 to 1 interval.

General EOQ model can not estimate the cost of existing ones, while using a fuzzy model can estimate the cost of lack of information that already exists . EOQ to be simplified modeling the EOQ model using fuzzy partial backorder and the principle of fuzzy membership functions. Partial backorder EOQ model is part deterministic EOQ model is a model which every ordering and inventory shortages can be known in advance. This paper will discuss the determination of the optimal amount of inventory model with partial backorder, backorder partial fuzzy EOQ models using the principle of triangle and trapezoid membership functions.

2. Main Results

2.1. Inventory Model with Backorder Using Partial

Assumptions used in the partial backorder EOQ model are goods ordered and kept only one kind, inventory immediately comes, and partial backorder allowed**[1]**.
Notation used in this model are:

TRC = Total Relevant Cost
t_1 = Inventory depletion time period
t_2 = Inventory backorder time period
R = Total annual requirement
V = Maximum Inventory
Q = Number of units each order
C = Order cost

H = Holding cost
K = Backorder cost
L = Loss cost
δ = Number of backorder units

In this discussion the total relevant costs for this model comes from order cost, holding cost, backorder costs, and losses sales cost.
For backorder unit is denoted by δ. In the model there are two phases of the inventory cycle namely: when the inventory depletion (t_1) and when inventory backorder (t_2).
Value of t_1 and t_2 defined**[1]**:

$$t_1 = T\left(\frac{V}{Q}\right) \quad (1)$$

$$t_2 = T\frac{(Q-V)}{Q} \quad (2)$$

For order costs, holding costs, backorder costs and the loss cost is defined as**[1]**:

1. Order Cost:

$$order\ cost = C\frac{R}{Q} \quad (3)$$

If assumed C = order cost, R = total annual requirement, Q = number of unit each order.

2. Holding cost:

$$holding\ cost = H\frac{V^2}{2Q} \quad (4)$$

for H = holding cost, V = maximum inventory

3. Backorder cost:

$$backorder\ cost = K\frac{\delta(Q-V)^2}{2Q} \quad (5)$$

assumed K = backorder cost

4. Loss cost:

$$Loss\ cost = L\frac{(1-\delta)(Q-V)^2}{2Q} \quad (6)$$

assumed L = loss cost

thus the total relevance cost is**[5]**:

$$TRC = \frac{CR}{Q} + \frac{H\,V^2}{2Q} + \frac{K\,\delta\,(Q-V)^2}{2Q} + \frac{L(1-\delta)(Q-V)^2}{2Q} \quad (7)$$

In (7) still contained the parameter t_1 and t_2 which has a different dimension to the planning period requirement R, therefore the dimensions should be synchronized first. The authors obtained the total relevance cost as follows:

$$TRC = \frac{CR}{Q} + \frac{H\,V^2}{2Q} + \frac{K(\delta)^2}{2Q} + \frac{L(1-\delta)^2}{2Q} \quad (8)$$

The purpose of this model formulation is finding the optimal Q and V, therefore the total relevant cost become:

$$TRC = C\frac{R}{Q} + H\frac{(V)}{2Q} + K\frac{\delta^2(H+K(\delta)+L(1-\delta))}{2Q} + L\frac{(1-\delta)^2}{2Q} (9)$$

Requirement to minimize (9) are:

$$\frac{\partial TRC}{\partial Q} = 0 \; and \; \frac{\partial TRC}{\partial V} = 0$$

then theoptimal Q and V for δ = 1 are:

$$Q = \sqrt{\frac{2CR(H+K)}{HK}} \quad (10)$$

$$V = \sqrt{\frac{2CRK}{H(H+K)}} \quad (11)$$

and for δ = 0, as follows:

$$Q = \sqrt{\frac{2CR(H+L)}{HL}} \quad (12)$$

$$V = \sqrt{\frac{2CRL}{H(H+L)}} \quad (13)$$

2.2. Partial Backorder Inventory Models for The Function of Trapezoidal Fuzzy

Notation for the use of trapezoidal fuzzy model are:

$T\tilde{R}C$: Total relevant costs under fuzzy models
$\tilde{H}$: Fuzzy number for holding cost
$\tilde{C}$: Fuzzy number for order cost
$\tilde{R}$: Fuzzy number for the number of annual demand
$\tilde{K}$: Fuzzy number backorder cost
$\tilde{L}$: Fuzzy number for loss cost
$\otimes$: Fuzzy number for multiplication
$\oslash$: Fuzzy number for division
$\oplus$: Fuzzy number for sum
$\tilde{Q}$: Unit number of once orderby fuzzy model
$\tilde{V}$: The maximum number of inventory by fuzzy model
$\mu_A(x)$: The level of a member function value for x
w : The value of the membership function

By using (7), TRC for fuzzy model is described as follows **[3]**:

$$T\tilde{R}C = \frac{\tilde{C} \otimes \tilde{R}}{\tilde{Q}} + \frac{\tilde{H} \otimes \tilde{V}^2}{2\tilde{Q}} + \frac{\tilde{K} \otimes \delta(\tilde{Q} - \tilde{V})^2}{2\tilde{Q}} + \frac{\tilde{L} \otimes (1 - \delta)(\tilde{Q} - \tilde{V})^2}{2\tilde{Q}}$$

Principle use of trapezoidal functions $\tilde{A}$ denoted by (a_1, a_2, a_3, a_4; w) where $a_1 \le a_2 \le a_3 \le a_4$is the principle function. Fuzzy calculation operation for two trapezoidal member functions $\tilde{A}$ and $\tilde{B}$are **[3]**:

$$\tilde{A} \oplus \tilde{B} = (a_1 + b_1. a_2 + b_2, a_3 + b_3, a_4 + b_4 \,; w) \quad (14)$$

$$\tilde{A} \otimes \tilde{B} = (a_1 b_1 . a_2 b_2 . a_3 b_3 . a_4 b_4 ; w) \quad (15)$$

Optimal solutions obtained using the condition:

$$\frac{\partial T\tilde{R}C}{\partial Q} = 0 \; and \; \frac{\partial T\tilde{R}C}{\partial V} = 0$$

thus obtained:

$$\tilde{Q} = \sqrt{\frac{2\sum_{i=1}^{4} c_i r_i}{\sum_{i=1}^{4} h_i}} \sqrt{\frac{\sum_{i=1}^{4} h_i + \sum_{i=1}^{4} k_i \delta + \sum_{i=1}^{4} l_i (1-\delta)}{\sum_{i=1}^{4} k_i \delta + \sum_{i=1}^{4} l_i (1-\delta)}} \quad (16)$$

$$\tilde{V} = \sqrt{\frac{2\sum_{i=1}^{4} c_i r_i}{\sum_{i=1}^{4} h_i}} \sqrt{\frac{\sum_{i=1}^{4} k_i \delta + \sum_{i=1}^{4} l_i (1-\delta)}{\sum_{i=1}^{4} h_i + \sum_{i=1}^{4} k_i \delta + \sum_{i=1}^{4} l_i (1-\delta)}} \quad (17)$$

The authors used fuzzy membership function consists of a crisp value, thus for a trapezoidal function, $\tilde{C} = (c_1 = c_2 = c_3 = c_4 = C)$. The results of the $\tilde{Q}$, $\tilde{V}$and T$\tilde{R}$C are:

$$\tilde{Q} = \sqrt{\frac{2CR(H + \delta K + (1-\delta)L}{H.K(\delta).L(1-\delta)}}$$

$$\tilde{V} = \sqrt{\frac{2CR(\delta K + (1-\delta)L)}{H(H + \delta K + (1-\delta)L)}}$$

$$T\tilde{R}C = \frac{CR}{Q} + \frac{HV^2}{2Q} + \frac{K\delta\,(Q-V)^2}{2Q} + \frac{L(1-\delta)(Q-V)^2}{2Q} \quad (18)$$

2.3. Partial Backorder Inventory Models for Triangular Fuzzy Function

Notation for the use of triangular fuzzy model are:

T$\tilde{R}$C : Total relevance cost according to the fuzzy model
$\tilde{H}$: Fuzzy number for holding cost
$\tilde{C}$: Fuzzy number for order cost
$\tilde{R}$: Fuzzy number for annual demand
$\tilde{K}$: Fuzzy number untuk backorder cost
$\tilde{L}$: Fuzzy number for loss cost
$\otimes$: Fuzzy number for multiplication
$\oslash$: Fuzzy number for division
$\oplus$: Fuzzy number for the sum
$\tilde{Q}$: Unit number of once order according to fuzzy models
$\tilde{V}$Ṽ : The maximum amount of inventory on the fuzzy model
$\mu_A(x)$: Level of member function value for x
w : Value of the membership function

By using (7), TRC for fuzzy model is described as follows **[3]**:

$$T\tilde{R}C = \frac{\tilde{C} \otimes \tilde{R}}{\tilde{Q}} + \frac{\tilde{H} \otimes \tilde{V}^2}{2\tilde{Q}} + \frac{\tilde{K} \otimes \delta(\tilde{Q} - \tilde{V})^2}{2\tilde{Q}} + \frac{\tilde{L} \otimes (1 - \delta)(\tilde{Q} - \tilde{V})^2}{2\tilde{Q}}$$

Principle use of the triangle function $\tilde{A}$denoted by (e_1, e_2, e_3, ; w) where $e_1 \leq e_2 \leq e_3$is the principle function. Operation calculation for two fuzzy triangular member function of $\tilde{E}$ and$\tilde{F}$are **[3]**:

$$\tilde{E} \oplus \tilde{F} = (e_1 + f_1 . e_2 + f_2 , e_3 + f_3 \,; w) \quad (19)$$

$$\tilde{E} \otimes \tilde{F} = (e_1 f_1 . e_2 f_2 . e_3 f_3 \,; w) \quad (20)$$

.

Optimal solutions obtained using the condition:

$$\frac{\partial T\tilde{R}C}{\partial Q} = 0 \; and \; \frac{\partial T\tilde{R}C}{\partial V} = 0$$

thus obtained:

$$\tilde{Q} = \sqrt{\frac{2\sum_{i=1}^{3} c_i r_i}{\sum_{i=1}^{3} h_i}} \sqrt{\frac{\sum_{i=1}^{3} h_i + \sum_{i=1}^{3} k_i \delta + \sum_{i=1}^{3} l_l (1-\delta)}{\sum_{i=1}^{3} k_i \delta + \sum_{i=1}^{3} l_l (1-\delta)}} \quad (21)$$

$$\tilde{V} = \sqrt{\frac{2\sum_{i=1}^{3} c_i r_i}{\sum_{i=1}^{3} h_i}} \sqrt{\frac{\sum_{i=1}^{3} k_i \delta + \sum_{i=1}^{3} l_l (1-\delta)}{\sum_{i=1}^{3} h_i + \sum_{i=1}^{3} k_i \delta + \sum_{i=1}^{3} l_l (1-\delta)}} \quad (22)$$

The authors used fuzzy membership function consists of a crisp value, thus for the triangular function,$\tilde{C} = (c_1 = c_2 = c_3 = C)$. The results of the $\tilde{Q}$, $\tilde{V}$ and T$\tilde{R}$C are:

$$\tilde{Q} = \sqrt{\frac{2CR(H + \delta K + (1 - \delta)L}{H . K(\delta) . L(1 - \delta)}}$$

$$\tilde{V} = \sqrt{\frac{2CR(\delta K + (1 - \delta)L)}{H(H + \delta K + (1 - \delta)L)}}$$

$$T\tilde{R}C = \frac{CR}{Q} + \frac{HV^2}{2Q} + \frac{K\delta\,(Q-V)^2}{2Q} + \frac{L(1-\delta)(Q-V)^2}{2Q} \quad (23)$$

2.4. Analysis

Aspects of the costs associated with inventory Super Star mattress:

a. Super Star Mattress = Rp.700.000
b. Order Cost (C) = Rp. 59.000/ order
c. Holding Cost (H) = Rp. 52.000/unit
d. Backorder Cost (K) = Rp. 38.000/unit
e. Loss Cost (L) = Rp. 35.000/ unit
f. Number Requirement Usage / year (R)= 10272 pcs
g. Unit number of backorder / Least backlog (δ)= 50 %
h. Average usage/ week = 214 pcs / week

Here is a product of consumer demand data Super Star mattress from March to May 2011 are presented in Table 1**[4]**:

Table 1 Consumer Demand Data

Mattress Type	**Consumer Demand**
1	216
2	106
3	193
4	338
5	246
6	185
7	133
8	161
9	213
10	282
11	219
12	275
Total	2566
Average	213,83

2.4.1. Partial Backorder Inventory Model

By using data that has been obtained from the company to find the total relevant cost (TRC) and the optimal amount of inventory to be able to calculate the partial backorder models with reference to the previous formula, to calculate the TRC:

a. Step 1: calculate the Q and V
   The results of Q = 237,734
   The results of V = 98,04
b. Step 2: Substitute Q and V to (7) to produce a number of TRC:
   The results obtained for TRC is 5,131,110.63

2.4.2. Partial Backorder Inventory Model Using Trapezoidal Fuzzy Function

Notation and cost using fuzzy model is:

a. Super Star Mattress = Rp.700.000
b. Order cost ($\tilde{C}$) = Rp. 59.000 – 72000 order
c. Holding cost ($\widetilde{H}$) = Rp. 52.000 – 74000 / unit
d. Backorder cost ($\widetilde{K}$) = Rp. 38.000 – 59000 / unit
e. Loss cost ($\tilde{L}$) = Rp. 35.000 – 65000 / unit
f. Number Requirement Usage / year ($\breve{R}$)= 10272 – 12240 pcs
g. Unit number of backorder / Least backlog (δ) = 50 %
h. Average usage/ week = 214 pcs – 255 /week

To find value $\tilde{Q}$, $\tilde{V}$and$T\tilde{R}C$of partial backorder model *backorder* using the following steps:

a. Step 1: calculate the$\tilde{Q}$and$\tilde{V}$
   The resultsof $\tilde{Q}$ = 231,60
   The results of $\tilde{V}$ = 101,84

b. Step 2: Substitute$\tilde{Q}$and$\tilde{V}$to (18) for find$T\tilde{R}C$ value
The resultsobtained for $T\tilde{R}C$ = 6.381.506,20

At the time variable has a crisp value $\tilde{C}$ = ($c_1 = c_2 = c_3 = c_4$ = C), it is a crisp value determination. For example the annual demand, ordering cost, holding cost, backorder cost and the cost of lost sales, is determined from the average value, then the value would appear to R = 11203, C = 65 750, H = 62 750, K = 48500, L = 50000 . Trapezoidal fuzzy membership functions will be $\tilde{C}$= ($c_1 = c_2 = c_3 = c_4$ = 65750), $\tilde{R}$ = ($r_1 = r_2 = r_3= r_4$ = 11203), $\tilde{H}$ =($h_1 = h_2 = h_3= h_4$= 62750), $\tilde{K}$ = ($k_1 = k_2 = k_3 = k_4$ = 48500), $\tilde{L}$ = ($l_1 = l_2 = l_3 =l_4$= 50000).
$T\tilde{R}C$ with an average value are generated using fuzzy trapezoidal membership functions as follows:

a. Step 1: calculate$\tilde{Q}$ and$\tilde{V}$using equation (16) and (17).
The results of$\tilde{Q}$ = 231,062
Then value of$\tilde{V}$ = 101,606

b. Step 2: Determine$\tilde{Q}$ and$\tilde{V}$to (18) for obtained $T\tilde{R}C$:
The resultsof $T\tilde{R}C = 6.375.773,585$

To prove that the results of crisp values $T\tilde{R}C$ for trapezoidal fuzzy models will be equal to the model without backorder partial fuzzy function can be calculated using equation (7). TRC with the following information:

a. Step 1: determine Q and V valueusing equation(16) and (17).
The results of Q = 231,062
The results of V = 101,606

b. Step 2: Substitute$\tilde{Q}$and $\tilde{V}$ to (7) for obtained $T\tilde{R}C$:
Then the results of $TRC = 6.375.773,585$

2.4.3. Partial Backorder Inventory Model Using Triangular Fuzzy Functions

Partial backorder inventory models using triangular fuzzy membership function is another model that can be used to consider the lack of value of information costs required by the company for the next period.

a. Step 1: Calculation$\tilde{Q}$and$\tilde{V}$ valueusing equation (21) and (22)
The results of$\tilde{Q}$ = 231,63
Then results of$\tilde{V}$ = 101,61

b. Step 2: Substitute$\tilde{Q}$and$\tilde{V}$to (23) for obtained$T\tilde{R}C$
The resultsof $T\tilde{R}C$ = 6.402.157,80

For crisp value $\tilde{C}$ = ($c_1 = c_2 = c_3 = c_4$ = C), which is an irreducible value. For example the annual demand, ordering cost, holding cost, backorder cost and the cost of lost sales for the model of triangular fuzzy membership function is determined from the average value, then the value would appear to R = 11256, C = 65500, H = 63000, K = 48500, L = 50000. Fuzzy membership functions will be $\tilde{C}$= ($c_1 = c_2 = c_3$ = 65500), $\tilde{R}$ = ($r_1 = r_2 = r_3$= 11256), $\tilde{H}$ = ($h_1 = h_2 = h_3$= 63000), $\tilde{K}$ = ($k_1 = k_2 = k_3$ = 48500), $\tilde{L}$ = ($l_1 = l_2 = l_3$ = 50000).
$T\tilde{R}C$ with an average value generated using triangular fuzzy membership functions as of the previous formula is:

a. Step 1: Calculate$\tilde{Q}$and$\tilde{V}$using equation (21) and (22).
The results of$\tilde{Q}$ = 230,965
The results of$\tilde{V}$ = 101,336

b. Step 2: Substitute$\tilde{Q}$ and$\tilde{V}$to (23) for obtained T$\tilde{R}$C:
Then the results of $T\tilde{R}C = 6.384.220{,}93$

To prove that the results of crisp values $T\tilde{R}C$ for trapezoidal fuzzy models will be equal to the model without backorder partial fuzzy function can be calculated using equation (7). TRC with the following information:

a. Step 1: determine Q and V using equation (21) and (22)
The results of$\tilde{Q}$ = 230,965
The results of $\tilde{V}$ = 101,336

b. Step 2: Substitute$\tilde{Q}$and$\tilde{V}$to (23) to obtained T$\tilde{R}$C :
Then the results of $T\tilde{R}C = 6.384.220{,}93$

This shows that when the lack of information is to be considered to find out how much it costs to run the next time. $T\tilde{R}C$ can be proved also for partial backorder models using fuzzy models will be higher than the TRC models without fuzzy partial backorder due to the use value of a trapezoidal or triangular membership functions are determined to be greater or equal to the specified cost, this is a realistic $T\tilde{R}C$ to get the fuzzy model in order to consider the uncertainty due to the lack of information about the costs of existing ones. Fuzzy EOQ models can then be used to identify and estimate the likely costs that will occur for the next period.

3. Concluding Remarks

EOQ influence on inventory very closely then by using EOQ fuzzy model with triangular and trapezoidal membership functions using partial backorder will be different with EOQ model. Optimal Q and V values for the optimal fuzzy models will have an increase due to the use of trapezoidal and triangular membership functions have a different value depending on the requirements of each membership function value can be an increase or equal. Whereas without the fuzzy it only has one fixed value of the costs. Therefore, by using a fuzzy model can solve the company's problems in estimating the costs for the next period.

References

[1] Purnomo, H.D,Wee, Hui-Meng, and Chiu,Yu Fang, *Fuzzy Economic Order Quantity Model With Partial Backorder*,Penang,Malaysia ICMBSE(2012).

[2] Heilpern, S., “The Expected Value of A Fuzzy Number,” *Fuzzy Sets and Systems*, 47, 81-86 (1992).

[3] Kusumadewi, S.,*Teknik dan Aplikasinya: Logika Fuzzy*,Artificial Intelligence Publisher (2003).

[4] UKRIDA Data Collection and Prosessing, Available: http://www.ukrida.ac.id/.../jkunukr-ns-s1-2009-222005011-1923-karya_prima-chapter3 - Ukrida [August 8, 2012]

[5] Yamit, Z.,*Inventory Management*, Ekonisia Publisher, Economic Faculty UII,Yogyakarta(1999).

A GOAL PROGRAMMING APPROACH TO SOLVE VEHICLE ROUTING PROBLEM USING LINGO

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Abstract. Vehicle routing problem (VRP) which discusses a set of routes for some vehicles, starting and ending at a depot, serving a set of customers such that each customer must be visited once by exactly one vehicle, and having a time constrain is called vehicle routing problem with time window (VRPTW). This paper presents a goal programming approach to solve VRP, especially VRPTW. We have considered an objective function with four main goals: to maximize utilization of vehicle capacity, to minimize the total waiting time, to minimize the total cost to serve the customers and to maximize the number of served customers. The proposed model was implemented and has been solved numerically using LINGO software and the optimal solution is presented.

Key words and Phrases: goal programming, vehicle routing problem, LINGO.

1. Introduction

Vehicle Routing Problem (VRP) is a set of routes which formed to serve a set of customers using vehicles, starting and ending at a depot. In VRP, each customer must be visited once by exactly one vehicle. The route has to be designed such that the total demands of all customers must not exceed the capacity of the vehicle.

Based on Jolai and Aghdaghi [1], Sousa *et al* [2], Azi *et al* [3], if VRP has a time constrains on the periods of the day in which each customer must be visited, it is called Vehicle Routing Problem with Time Window (VRPTW). VRPTW is one of an important problem occurring in distribution systems. So, it has received many attentions not only on the development of the theory, but also on its application. For example, postal deliveries, delivery service of food business, Liquefied Petroleum Gas (LPG) deliveries. Routes which designed should be in a short duration and must be satisfy time constrain. Larsen [4], Cook and Rich [5], Cordeau *et al* [6] proposed an exact method for the VRPTW. As development of research, there are many other methods to solve VRPTW.

On the other side, because of its wide application to real-life situations, distribution problems have other objectives than minimizing the total travel time or distance. Multiple objectives of VRP can be found in Hong [7], Calvete [8], and Hashimoto [9]. The goal programming approach is a great method to solve the multiple objectives. The aim of goal programming is to minimize the deviations between the goals. Jolai and Aghdaghi [1], Hong [7], and Calvete [8] used goal programming to model the VRPTW.

In this paper we use goal programming approach to solve VRP, especially VRPTW. The remainder of this paper is organized as follow. In Section 2, we present the mathematical formulation of VRPTW which is described as a goal programming approach. In Section 3, we implemented the model using a set of data and solve it using LINGO. Finally, Section 4 concludes the paper.

2. Model Formulation

We formulate the problem as follow. Let $G\left(N^*, E\right)$ be a directed graph associated with the problem. We define $N = \{1, 2, ..., n\}$ be a set of nodes such that each representing a customer location. Notation $N^* = \{0, 1, 2, ..., n, n+1\}$ is the set of nodes, where node 0 represents the depot, and node $n+1$ refers to a copy of depot. We have notation $E = \{(i, j) : i, j \in N^*\}$ as the set of directed arcs (potential route between the customers and the depot).

There is a known demand d_i for every customer $i \in N$. The sum of the demands of the customers served by a designed route cannot exceed *q,* where *q* is the capacity of vehicle that used for delivering goods from the depot to the customers. We also define s_i as a known service time for each customer *i.*

Each arc $(i, j) \in E$ has an associated travel cost c_{ij} and a travel time t_{ij}. It represents cost and time of going from node *i* to node *j* through arc (i, j). Node 0 and node $n+1$ are only incident to outgoing and incoming arcs, respectively. The set of route denoted by $R = \{1, 2, ..., r\}$, and it should be noted that the route taken by a vehicle have to depart from node 0 and terminate at node $n+1$. For each $k \in R$, we define C_k as the travel cost to serve the customers at route *k*, and we use notation T_k as a distribution time of all customers in route *k.* So, we can assume each customer in a route has to be served before $T_{\max}$ with total cost less than $C_{\max}$.

To formulate the model, we define the following variables:

Decision variables:

$$x_{ij}^k = \begin{cases} 1, & \text{if there is a vehicle travel from node i to node j in a route k} \\ 0, & \text{otherwise} \end{cases}, \tag{1}$$

$$y_i^k = \begin{cases} 1, & \text{if node i is visited in route k} \\ 0, & \text{otherwise} \end{cases}, \tag{2}$$

w_i^k is the time of beginning of service at node *i* in route *k*

In this paper we use goal programming approach for the problem which has four main goals. First goal, maximize utilization of vehicle capacity. Second, minimize total distribution time of all customers. Third, minimize the total travel cost to serve the customers, and forth goal, maximize the number of served customers. So, we define deviational variables of each goal.

Deviational variables:

$\sum_{k\in R}\omega_{1k}^{-}$ is negative deviational variable of first goal,

ω_2^{+} is positive deviational variable of second goal,

ω_3^{+} is positive deviational variable of third goal,

$\sum_{i\in A}\omega_{4i}^{-}=\begin{cases}1, & \text{if customer i can be served}\\ 0, & \text{otherwise}\end{cases}$, it is negative deviational variable of forth goal.

The purpose of goal programming is to minimize the given deviations between the goals. From the goals that has presented before and variables we define, if $\lambda_1,...,\lambda_4$ as weight of each deviational variable, respectively, so the objective function of the problem can be written as follow:

$$\min Z=\lambda_1\sum_{k\in R}\omega_{1k}^{-}+\lambda_2\omega_2^{+}+\lambda_3\omega_3^{+}+\lambda_4\sum_{i\in A}\omega_{4i}^{-}, \tag{3}$$

The problem has some constrains. We will explain one by one as follow:

Because the sum of the demands of the customers served at route *k* cannot exceed *q,* and represent from the first goal by using negative deviational variable, so we have

$$\sum_{i\in N}d_i y_i^k+\omega_{1k}^{-}=q, \qquad \forall k\in R. \tag{4}$$

The distribution time in route *k* is derived from sum of total service time and total travel time in route *k*. Using notation T_k as distribution time in route *k,* so

$$T_k=\sum_{i\in N}s_i y_i^k+\sum_{i\in\{0\}\cup N}\ \sum_{j\in N\cup\{n+1\}}t_{ij}x_{ij}^k, \qquad \forall k\in R. \tag{5}$$

We should be noted that total distribution time have to less than the given time, $T_{\max}$, so

$$\sum_{k\in R}T_k\le T_{max}. \tag{6}$$

Related to the first goal, in order to guarantee that total distribution time is minimize by using positive deviational variable, so

$$\sum_{k\in R}T_k-\omega_2^{+}=0. \tag{7}$$

Travel cost to serve the customers at route *k* is sum of travel cost such that there is a vehicle going from node *i* to node *j* through arc (i, j) in route *k*. Using notation C_k as travel cost to serve the customers at route *k*, then

$$C_k=\sum_{(i,j)\in E}c_{ij}x_{ij}^k, \qquad \forall k\in R. \tag{8}$$

But, total travel cost should less than $C_{\max}$

$$\sum_{k\in R} C_k \le C_{\max} . \tag{9}$$

According to the third goal, we define Eq. (10) to guarantee that total travel cost is minimize by using positive deviational variable, that is

$$\sum_{k\in R} C_k - \omega_3^{+} = 0 .$$

(10)

If customer *i* can be served at route *k,* then should be there is a vehicle travel from customer *i* to the other customer in route *k.* It means, we have to guarantee each customer can only visited once

$$\sum_{j\in N^*} x_{ij}^k = y_i^k, \qquad \forall i \in N^*, k \in R . \tag{11}$$

Refer to the forth-goal, each customer can only be visited once, and from the objective function, sum of the negative deviational variable $\sum_{i\in A} \omega_{4i}^{-}$ which is weighted by λ_4 computes the number of customers who have not been served, then in order we can maximize the number of served customers

$$\sum_{k\in R} y_i^k + \omega_{4i}^{-} = 1, \qquad \forall i \in N . \tag{12}$$

To ensure that all routes leave and return to the depot, so we write

$$\sum_{j\in N} x_{0j}^k = 1, \qquad \forall k \in R , \tag{13}$$

$$\sum_{i\in N} x_{i(n+1)}^k = 1, \qquad \forall k \in R . \tag{14}$$

For every vehicle that has visited customer *i,* it means the vehicle will leave customer *i,*

$$\sum_{i\in\{0\}\cup N} x_{i\ell}^k - \sum_{j\in N\cup\{n+1\}} x_{\ell j}^k = 0, \qquad \forall \ell \in N, \forall k \in R . \tag{15}$$

If w_i^k is a notation for the start time of service at customer *i* of route *k,* then we can guarantee feasibility of the time schedule by

$$w_i^k + s_i + t_{ij} - M\left(1 - x_{ij}^k\right) \le w_j^k, \qquad \forall i, j \in N^*, \forall k \in R , \tag{16}$$

where *M* an arbitrary large constant. Each customer *i* has a time window $\left[\alpha_i, \beta_i\right]$ meaning the vehicle must arrive during the interval, so the start time of service at customer *i* should be in the interval

$$\alpha_i \le w_i^k \le \beta_i, \qquad \forall i \in N, \forall k \in R , \tag{17}$$

Eq. (3) subject to (4) – (17) is called goal programming model of the VRPTW. In the next section, we will discuss how to solve this problem using LINGO.

3. Computational Result

Now we present some tests and results to a real problem of the Liquefied Petroleum Gas (LPG) agent. The LPG will be distributed to the customers. In this paper we only take the first 5 customers as an example. We denote each customer by N1, N2,…, N5. For depot and copy of depot will be denoted by N0 and N6, respectively. The travel time, travel cost and the demand of the customers are presented in Table 1, Table 2, and Table 3.

Table 1. Travel time between nodes (minutes)

	Depot	N1	N2	N3	N4	N5
Depot	0	2	3	3	5	3
N1	2	0	1	4	5	4
N2	3	1	0	5	6	5
N3	5	4	5	0	5	5
N4	5	6	7	3	0	5
N5	3	7	8	6	7	0

Table 2. Travel cost between nodes (x Rp 1000,00)

	Depot	N1	N2	N3	N4	N5
Depot	0	0.27	0.3	0.41	0.54	0.38
N1	0.25	0	0.06	0.48	0.67	0.48
N2	0.28	0.06	0	0.45	0.70	0.51
N3	0.45	0.41	0.45	0	0.54	0.41
N4	0.54	0.67	0.70	0.35	0	0.51
N5	0.67	0.77	0.80	0.64	0.57	0

Table 3. Demand and service time for each customer

Customer	N1	N2	N3	N4	N5
Demand (units)	70	420	80	120	240
Service time (minutes)	23	140	27	40	80

Complete script of the problem solved in LINGO 11.0 shown bellow

```
MODEL:
 SETS:
N/N0,N1,N2,N3,N4,N5,N6/:q,s,v,h,d2;
R/R1..R2/:B,TR,d4;
E(N,R):Y,W;
D(N,N,R):X;
A(N,N):c,t;
ENDSETS
DATA:
U = 56;
UC = 3;
UT = 6;
q = 0 7 42 8 12 24 0;
s = 0 23 140 27 40 80 0;
 !0;!1;!2;!3;!4;!5;!10;
c = !0;0.0.270.300.410.540.380.
 !1;0.250.0.060.480.670.480.27
 !2;0.280.060.0.450.700.510.30
 !3;0.450.410.450.0.540.410.41
 !4;0.540.670.700.350.0.510.54
 !5;0.670.770.800.640.570.0.38
 !10;0.0.270.300.410.540.380.   ;

 !0;!1;!2;!3;!4;!5;!10;
t =!0;0233530
 !1;2014542
 !2;3105653
 !3;5450555
 !4;5673055
 !5;3786703
 !10;0233530    ;
v = 0 1 1 2 1 1 0;
h = 0 1 1 2 1 1 0;

ENDDATA
min = d1 + @sum(N(I)|I#NE#1 #AND# I#NE#7:d2) + d3 + @SUM(R(K):d4);
@FOR(R(K):@SUM(N(I):q(I)*Y(I,K))+d4(K)=U);
@FOR(R(K):(@SUM(A(I,J):t(I,J)*X(I,J,K))+@SUM(N(I):s(I)*Y(I,K)))/60=TR);
@SUM(R(K):TR)<=UT;
@SUM(R(K):TR)-d3=0;
@FOR (R(K):B=@SUM(A(I,J):c(I,J)*X(I,J,K)));
@SUM(R(K):B)<=UC;
@SUM(R(K):B)-d1=0;
@FOR(R(K):@FOR(N(I):@SUM(N(J):X(I,J,K))=Y(I,K)));
@FOR(N(I)|I#NE#1#AND#I#NE#8:@SUM(R(K):Y(I,K))+d2=1);
@FOR(R(K):@SUM(A(I,J)|J#EQ#7:X(I,J,K))=1);
@FOR(R(K):@SUM(A(I,J)|I#EQ#1:X(I,J,K))=1);
@FOR(R(K):@FOR(N(I):@SUM(N(J)|I#NE#1 #AND# I#NE#7:X(I,J,K))-
@SUM(N(J)|I#NE#7 #AND# I#NE#1:X(J,I,K))=0));
@FOR(R(K):@FOR(N(I):@FOR(N(J):(W(I,K)+S(I)+t(I,J))-100000*(1-
X(I,J,K))<=W(J,K))));
@FOR(N(I):@FOR(R(K): v*Y(I,K) <= W(I,K)));
@FOR(N(I):@FOR(R(K): W(I,K) <= h*Y(I,K)));
@FOR(D(I,J,K):@BIN(X));
@FOR(R(K):@BIN(Y));
@FOR(N(I):@BIN(d2));
END
```

Fig. 2. Complete Script Using LINGO 11.0

By inputting vehicle capacity 560 units, maximum distribution time 6 hours a week, and maximum distribution cost Rp 3.000,00 a week, we get two route which summarize in Table 4.

Table 4. Summarize output of LINGO

Route	Total Distribution Cost	Total distribution time (hour)	Total unit LPG
N0-N2-N1-N6	Rp 610,00	2.82	490
N0-N4-N3-N5-N6	Rp 1.970,00	2.77	440
Sum	Rp 2.580,00	5.59	930

From Table 4, all customers can be served by the agent. If we decrease the maximum distribution time to 5 hours a week, then only 4 customers can be served. It is also logic, if we decrease to 4 hours a week, then only 2 customers can be served. Decreasing the number of customers which can be served also happened whenever we decrease maximum distribution cost to Rp 2.000,00. It is only 4 customers can be served.

4. Conclusion

In this paper, we proposed a goal programming approach to solve Vehicle Routing Problem with Time Windows (VRPTW). Eq. (3) subject to (4) – (17) is called goal programming model of the VRPTW. The model was implemented and the results had been obtained using LINGO. From the simulation, by increasing the value of $T_{\max}$, it is followed the increasing customers can be served. As a future research, we suggest improving our model to solve larger nodes.

References

[1] Jolai, F., & Aghdaghi, M. (2008). A Goal Programming Model for SIngle Vehicle Routing Problem with Multiple Routes. *Journal of Industrial and Systems Engineering* , 154-163.

[2] Sousa, J. C., Biswas, H. A., Brito, R., & Silveira, A. (2011). A Multi Objective Approach to Solve Capacitated Vehicle Routing Problems with Time Windows Using Mixed Integer Linear Programming. *International Journal of Advanced Science and Technology* , 1-8.

[3] Azi, N., Gendreau, M., & Potvin, J.-Y. (2007). An exact algorithm for a single-vehicle routing problem. *European Journal of Operational Research* , 755-766.

[4] Larsen, J. (1999). *Parallelization of the Vehicle Routing Problem with Time Windows.* Institute of Mathematical Modelling. Denmark: Technical University of Denmark.

[5] Cook, W., & Rich, J. (1999). *A Parallel Cutting-Plane Algorithm for the Vehicle Routing Problems with Time.* Department of Computational and Applied Mathematics. Houston: Rice University.

[6] Cordeau, J.-F., Gendreau, M., Laporte, G., Potvin, J.-Y., & Semet, F. (2002). A guide to vehicle routing heuristics. *Journal of the Operational Research Society* , 512-522.

[7] Hong, S., & Park, Y. (1999). A heuristic for bi-objective vehicle routing with time window constraints. *International Journal of Production Economics* , 249-258.

[8] Calvete H.I., G. C. (2007). A Goal Programming Approach to Vehicle Routing Problems With Soft Time Windows. *European Journal of Operational Research* , 1720-1733.

[9] Hashimoto H., I. T. (2006). The Vehicle Routing Problem With Flexible Time Windows and Travelling Times. *Discrete Applied Mathematics* , 1364-1383.

CLUSTERING SPATIAL DATA USING AGRID+

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Abstract. Clustering is a branch of data mining which focus to form clusters or groups from data. Many methods has been developed to accomplish this task, one of them is AGRID+. In this paper, we proposed a new approach of clustering that is using simplify AGRID+ to cluster spatial data. The idea is to separate the location attribute from the main attribute. We form the cell based on location attribute and calculate the proximity of two objects based on the main attribute. Experimental result that are using simulated data and criminal data from POLDA METRO JAYA area are reported.

*Key words and Phrases:*Clustering, AGRID+, Spatial Data, Grid Clustering, Criminal Data

1. Introduction

Clustering is one of the main techniques in data mining that focus to form clusters (or groups) from data where objects that belong to same cluster are similar or close to each other, and objects that not in same cluster are not similar. Many methods and algorithms has been developed to solve this problem. Most clustering methods fall into four categories : hierarchy based, partition based, grid based, and density based[4]. Grid based algorithms apply a grid structure into data which then quantize the data into finite number of cells. The benefit of this technique is it's fast processing time so this approach is suitable for handling a large data. One of the grid based clustering algorithm is AGRID+[13]. AGRID+ combines grid based and density based approach to clustering large data with complex shape, although the experiment was done only using normal data that need no special treatment.

In clustering, there are some kind of data that can be clustered. One of them is spatial data. Spatial data is a data that have not only the main attribute (normal attribute) but also contain the location attribute of the data. This attribute differ from main attribute because its function is not to measure the similarity between objects. With spatial attribute, we can say whether an object located in north of another object or not. There is not much algorithms than can handle this kind of data because it cannot be clustered directly and need special treatment. In this paper, we formulate a new approach to clustering spatial data using AGRID+.

2. Related Work

Some clustering algorithms of spatial data are the results of an extension from previous clustering algorithms[15][18]. Therefore, the categorization of spatial data clustering algorithms follows the categorization of common clustering algorithms. The conversion of common clustering algorithms into spatial clustering algorithms is done by involving spatial elements in the clustering algorithms. Each clustering algorithm has its own unique way of involving spatial elements. The following paragraph presents some spatial clustering algorithms which are the extensions of usual clustering algorithms.

Partition method divides data into subsets or partitions by evaluating the distance between data and cluster representation. Some proposed and developed algorithms which are based on partition method are k-means, k-medoid, fuzzy k-means, PAM, CLARA, and CLARANS[19][20][23]. CLARANS algorithm enhances PAM and CLARA algorithms in terms of their computation efficiency. Two methods of clustering which adopt CLARANS algorithm for spatial data are termed spatial dominant or SD approach (CLARANS) and non-spatial dominant or NSD approach (CLARANS)[20][22]. CLARANS algorithm has also been implemented for spatial data in the form of Raymond's polygon[20].

Some algorithms that perform clustering process based on density are DBSCAN, DENCLUE, and OPTICS[23]. Sander extended DBSCAN by involving spatial elements and non-spatial elements simultaneously to cluster spatial data in 2 to 5 dimensional format. DBCLASD is another extension of DBSCAN in spatial clustering algorithm. Likewise, the density-based clustering algorithm developed by Yang et.al. (2010) and Liu et.al. (2012) was extended by using Delaunay triangulation[20][17].

3. AGRID+ for Spatial Data

There are 4 main features of AGRID+. First, instead of cells, objects are taken as the smallest units. Second is the concept of *i*th-order neighbors, where neighbor cells are organized into a couple of groups. Third is density compensation to improve accuracy. The last feature is new distance measure, *minimal subspace* distance for subspace clustering.

In grid based clustering algorithms, the concept of neighbor cell is very important. There two concept of neighbor cell i.e. *full neighbor* and *immediate neighbor*. The accuracy of full neighbor is high because its consider all cell that around the main cell. But the computation is quite high too, especially with the increase of dimensionality of the data. To solve this problem, new neighbor concept is introduced, that is the *i*th-order neighbors, where cells are grouped based on its contribution. With this concept, we don't need to consider all cells, but only those who have high contribution.

With the concept of *i*th-order neighbors, not all cells are considered. Then the result is not as good as when we use the full neighbor. Density compensation is introduced to handle this problem. With density compensation, the ratio between volume of used cell and volume all cell are considered. With this technique, the accuracy of AGRID+ will increase even if we didnt use the full neighbor.

With that features, AGRID+ can found clusters with different shape and with its grid technique, it can clustering large data or data with high dimension more efficiently. The procedure of AGRID+ consist 7 main steps.

1. *Partitioning*. Data space is partitioned into cells. Each object then are labeled to its cell according to its attribute.

2. *Computing distance threshold*. The distance threshold is computed based on interval each dimension.
3. *Calculate density*. For every object, count the number of object in its neighbor cells that the distance between both is less than distance threshold.
4. *Density compensation*. For each object, count the ratio between all cells and the number of cells used. This ratio multiply with the object density and save as the new density for the object.
5. *Compute Density Threshold (DT)*. DT is the average of all compensated density.
6. *Clustering*. First, each object whose density greater than DT is taken as a cluster. Then, for every object in its neighbor whose its density greater than DT too, if the distance between those two less than distance threshold, then merge those two cluster. Continue the merging process until all object have been checked.
7. *Removing outliers*. Some cluster might only have few objects in it. They too small to be considered as a meaningful clusters, so they are removed and labelled as noises.

In this paper, there are some modified of AGRID+ algorithm. There are two modification, i.e first, density compensation, instead of volume, we used the number of cells as the ratio. Second, the dissimilarity measured is used euclidean distance as the distance measure instead of minimum subspace distance of AGRID+. That's way we called simplified AGRID+.

Our proposed is to cluster spatial data using AGRID+. Using AGRID+ directly to cluster spatial data, it will treat the spatial attribute of an object same as the main attribute. Therefore the clusters that exist in main attribute are distorted. So, the proposed method is using spatial attribute to determine the cells (grid). Clustering method done in grid that developt from spatial attributes of the data set.

3. Experiment and Analysis

To get the real result from spatial clustering, we have to treat the spatial attribute differently. In our method, we separate the spatial attribute from the main attribute. We used the spatial attribute only to form cells, and we used the main attribute to measure similarity between two objects.

We have done 4 experiments using this method. In the first two experiments, we used data that we generate randomly. We set this data to have cluster in spatial and main attribute so we can see whether the method can found the cluster based on main attribute only and not affected by spatial aspect of the object. For the last two experiments, we used crime data. The crime data used is crime data of under POLDA METRO JAYA jurisdiction area (Jakarta, Depok, Bekasi, Tangerang) from year 2009 to 2012. In each experiment, the result were taken based on its evaluation. We combined some parameters to get the best result.

The first experiment was done using simulation data with the number of main attribute is two and the number of objects are 1000. The data were set so there only two clusters exist in one spatial area. The clustering result can be seen in figure 1. The data clustered into two clusters although in spatial space, it only has one cluster. From this experiment, we can see that our method can detect the cluster very well without being affected by spatial aspect of the object. We did the second experiment using 2000 data that we set into three clusters where spatially form two spatial areas. We also set one of the cluster to exist in both spatial area. This method can found all the clusters, including cluster that exist in both spatial area (see figure 1).

From the figure, we can see that using our approach, spatial attribute from the object is only used for determine the neighbor of an object and therefore not related to the measure of similarity of the object.

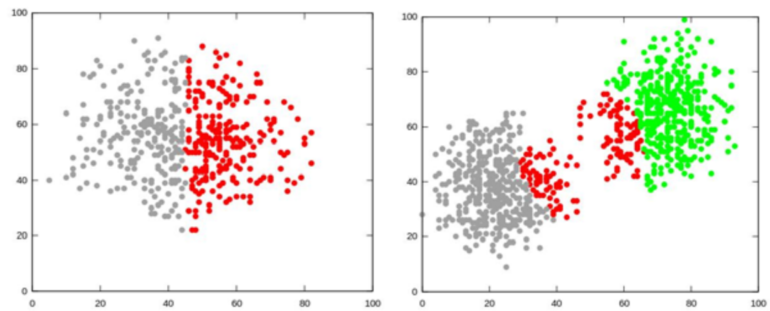

Figure 1:Clustering result using 2D generated data

The next two experiments were done using crime data from Jakarta, Depok, Tanggerang, Bekasi, Kabupaten Tanggerang and Kabupaten Bekasi from 2009 to 2012. There are 16 kinds of crime, that is murder, mayhem, thievery, hold up crime, raiding, hijacking, 2 wheels vehicle, 3 wheels vehicle, 4 wheels vehicle, fire, gambling, extortion, rape, narcotics, and Juvenile Delinquency. But not all kind crime were used, we only used a few kind (as attribute) that look more "interesting". Every object is a representation of each district in each region. The spatial component were taken approximately using grid technique.

The third experiment was done using only the narcotics attribute. To get the result, we combine some parameters and then we take the best result based on its evaluation index. In our experiment, we used silhouette evaluation method (SH) to evaluate the clustering result[10]. In SH, the higher the index is (get close to 1) means the more good the result is. The clustering result and its evaluation index can be seen in table 1.

Table 1. Clustering Result using narcotics attribute

Year	SH	Ratio (%)	Clusters
2009	0.31005	22.33	2
2010	0.3609	35.922	4
2011	0.822	13.59	2
2012	0.529	12.417	2

From the table above, we can see that the evaluation result using our method is not quite high. Only in 2011 that SH shows value above 0.7, the rest falls mainly between 0.3 – 0.5. And also as we can see, the number of object that are clustered is quite small too, where the ratio of clustered object still under 35%. The result of clustering in 2010 can be seen in figure 2.

Figure 2:Clustering result in 2010 using narcotics attribute

The last experiment was done using 3 attributes. In this experiment we combine some attribute into one. The first we combine mayhem, hold up crime, raiding, and hijacking attribute into one attribute, theft with violence. And also all vehicle crime (2 wheels, 3 wheels and 4 wheels) into one vehicle crime. The last attribute is thievery. The experiment scenario still same, that is we combine several value of parameter and find the best result. The best result still based on SH evaluation index. Table 2 shows the clustering result from 2009 to 2012.

Table 2: Clustering result of thievery, theft with violence, and vehicle attributes

Year	SH	Ratio (%)	Clusters
2009	0.139	22.33	2
2010	0.602	18.446	2
2011	0.378	17.475	2
2012	0.460	19.417	2

Clustering result in 2010 can be seen in figure 3.

Figure 3:Clustering result in 2010 using 3 attributes

3. Concluding Remarks

In our research, we propose a new approach to clustering spatial data. The result of our simulation data shows that our method can found the two clusters which in spatial space only have one cluster. For real data, the result of our experiment shows in some year, data are well clustered. This can be shown with the value of evaluation index that greater than 0.7. But since we used silhouette (SH) index for the evaluation and the SH is not designed to evaluate spatial data, the result might be not optimal. This is our future work to provide an evaluation technique that can evaluate spatial data.

References

[1] Brian S. Everitt, Sabine Landau, M. L. D. D. S. *Cluster Analysis*, 5 ed. Probability and Statistics. John Wiley & Sons, 2011.

[2] Dunham, M. *Data Mining: Introductory And Advanced Topics*. Pearson Education, 2006.

[3] Gholamhosein Sheikholeslami, Surojit Chatterjee, A. Z. *Wavecluster : a wavelet-based clustering approach for spatial data in very large databases*. VLDB Journal 8 (2000), 289 – 304.

[4] Guojun Gan, Ghaoqun Ga, J. W. *Data Clustering Theory, Algorithms, and Applications*. ASA-SIAM Series on Statistic And Applied Probability. ASA, 2007.

[5] Jiawei Han, Micheline Kamber, J. P. *Data Mining Concepts and Technique*, 3 ed. Data Management System. Morgan Kaufmann, 2012.

[6] Kantardzic, M. *Data Mining : Concepts, Models, Methods, and Algorithms*. Johw Wiley & Sons, 2003.

[7] Kromesch, S., and Juhsz, S. *High dimensional data visualization*.

[8] Larose, D. T. *Discovering Knowledge In Data : An Introduction to Data Mining*. John Wiley & Sons, 2005.

[9] Mr Ilango, D. V. M. *A survey of grid based clustering algorithms*. International Journal of Engineering Science and Technoloy 2 (2010), 3441–3446.

[10] Rousseeue, P. J. *Silhouettes : a graphical aid to the interpretation and validation of cluster analysis*. Journal of Computational and Applied Mathematics 20 (1987), 53–65.

[11] Tan, P.-N., Steinbach, M., and Kumar, V. *Introduction to DataMining*, us ed. Addison Wesley, May 2005.

[12] Wei Wang, Jiong Yang, R. M. *Sting : A statistical information grid approach to spatial data mining*. VLDB Conference volume 23 (1997).

[13] Yanchang, Z., and Junde, S. Gdilc : *A grid-based density isoline clustering algorithm*. IEEE 3 (2001), 140–145.

[14] Zhao, Y., Cao, J., Zhang, C., and Zhang, *S. Enhancing grid-density based clustering for high dimensional data*. Journal of Systems and Software 84, 9 (2011), 1524 – 1539.

[15] A. Varlaro, "Spatial Clustering of Structured Objects," University of Bari, Italy, 2008.

[16] X. Yang and W. Cui, "A Novel Spatial Clustering Algorithm Based on Delaunay Triangulation," J. Software Engineering & Applications, vol. 2010, no. February, pp. 141–149, 2010.

[17] Q. Liu, M. Deng, Y. Shi, and J. Wang, "A Density-based Spatial Clustering Algorithm Considering Both Spatial Proximity and Attribute Similarity," Computers and Geosciences, Elsevier, vol. 46, pp. 296–309, 2012.

[18] S. Shekhar and S. Chawla, Spatial Databases A Tour. New Jersey: Prentice Hall, 2003

[19] L. Kaufman and P. J. Rousseeuw, Finding Groups in Data: An Introduction to Cluster Analysis. New Jersey: John Wiley & Sons, 1990, pp. 1–355.

[20] N. Raymond T. and J. Han, "CLARANS : A Method for Clustering Objects for Spatial Data Mining," IEEE transactions on Knowledge and Data Engineering, vol. 14, no. 5, pp. 1003–1016, 2002.

[21] Rama.B, Jayashree.P, and S. Jiwani, "A Survey on clustering," International Journal on Computer Science and Engineering, vol. 02, no. 09, pp. 2976–2980, 2010.

[22] F. Zhou and S. K. Zhan, "Analysis of spatial clustering of disease," Chinese journal of preventive medicine, vol. 28, no. 6, pp. 337–339, 1994.

[23] M. Ester, H. Kriegel, J. Sander, and X. Xu, "A Density-Based Algorithm for Discovering Clusters in Large Spatial Databases with Noise," in 2nd International Conference on Knowledge Discovery and Data Mining (KDD-96), 1996.

CHAOS-BASED ENCRYPTION ALGORITHM FOR DIGITAL IMAGE

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Abstract. Today, storage and transmissions of data or information is unproblematic by supports of information and communication technology. Data or information presented in digital form is highly vulnerable by attack of data or information abusing. Digital image is one of digital data or information which is frequently becomes in target crime. Therefore, reliable, secure, and fast security technique is required in data or information digital image. In this study, encryption algorithm is built using logistic map as a random number generator. This algorithm applied in digital image. Designing in chaos-based encryption algorithm is improved endurance from brute force and known plaintext attack. Performance on algorithm endurance was based on key space analysis, sensitivity analysis to the initial values, and histogram analysis using a goodness-of-fit test. According to testing and analysis, this algorithm has a key space of 10^{30} and sensitivity to initial values is very sensitive, up to 10^{-16}. It can be concluded that, the algorithm is very difficult to be cracked by brute force attack.

Key words and Phrases: Chaos, Logistic map, Encryption algorithm, Digital image.

1. Introduction

Until the 1990s the encryption algorithm that is often used is DES, but because of the advanced technology, these algorithms are not considered safe anymore. Therefore, an international competition held by the National Institute of Standards and Technology (NIST)and Rijndael algorithm was selected as the winner after passing through various stages of the selection in 2001, which was later renamed AES algorithm [4].The performance of an algorithm can be seen from the resistance against attacks and security algorithms computational time. Traditional ciphers such as Data Encryption Standard (DES), Advanced Encryption Standard (AES), and Rivest-Shamir-Adleman Algorithm (RSA) etc. to encrypt image requires a large computational time and high computing power. But, only thoseciphers are preferable which take lesser amount of time and atthe same time without compromising security [2].

To provide a better solution from the image security issues, a number of image encryption techniques have been proposed, one of which is a chaos -based image encryption. This method gives a good combination of speed, high security, complexity , and computational power [3].

Chaos is the type of behavior of a system or function that is random, sensitive to initial values, and ergodicity. Function that has chaos propertieswas called chaos function. Chaos function have been proved very suitable to design facilities for data protection [1]. With these properties, chaos function can be used as a random number generator. One of the simple function that shows the chaos properties is the logistic equation or commonly called the logistic map. Other functions that have chaos properties are Henon chaotic map, Arnold 's cat map, and the tent map. Logistic map function is defined as a function :

$$L_\lambda : R \rightarrow R , L_\lambda (x) = \lambda x (1 - x)$$

which is a function of two variables λ and x. λ variable value in the interval $(0,4]$ and x are in the interval $[0,1]$. Meanwhile, the presentation of logistic map function is in the form of iterative. It is :

$$x_{n+1} = \lambda x_n (1 - x_n) \qquad (1)$$

with $n = 0 , 1,2,3 \ldots.$ and x_0 is the initial value of iteration .

For the last decade , a lot of chaos -based cryptographic techniques have been studied, such as secret communication based on chaos, chaos based block / stream cipher,chaos -based random number generator, etc. As for the chaos -based security applications such as image encryption and chaos based copyright protection of multimedia [1]. Chaos-based encryption also been extensively studied by researchers because of its superior in safety and complexity [6].Therefore, in this paper. we will discuss about security of digital imageusing chaos -based encryption method , by using the logistic map as a chaos function.

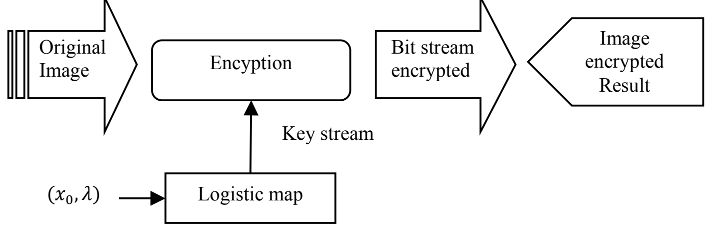
Testing of algorithm was done based on key space analysis, key sensitivity analysis, the encryption average time. The analysis was carried out to see the resistance againtsbrute force attack and known plaintext attack .

2. Main Results

2.1 Algorithm

Digital image encryption algorithm in this paper uses logistic map as a chaos function (equation 1). The sequence of the process of securing the digital image can be seen in Figure 2.1 below :

A.



B.

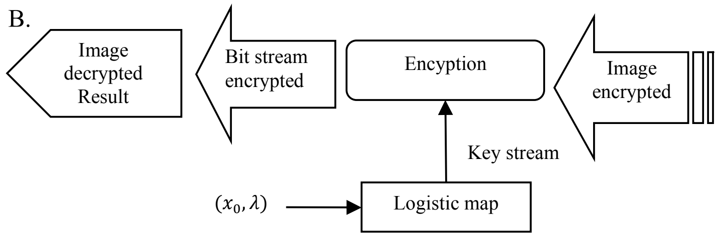


Figure 2.1. The sequence of the process of securing the digital image
A.) Encryption process B.) Decryption process

Figure 1 shows the flow in securing digital images. On A, first input the original image and then encrypt it using logistic map as a key stream generator, where logistic map needa key(x_0, λ) and from that we obtain the bit stream encrypted, that is the image result. On B , the reverse process from encryption process on A, with using the same key.

Encryption algorithm is described in the flowchart shown in Figure 2.2:

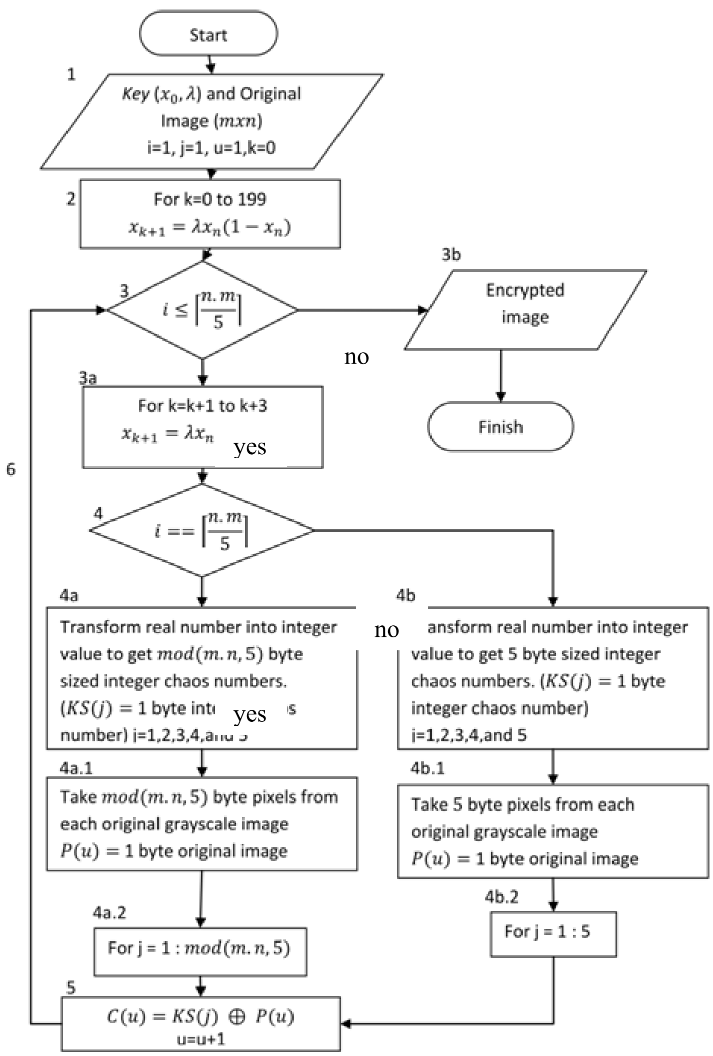


Figure 2.2 Flowchart of digital image encryption algorithm using the logistic map

Flowchart explanation is shown in Figure 3.4 *:*

Step 1 : Insert the*key*(x_0, λ)andoriginal imagewith*nxmsize*

Step 2 : Do 200 times iteration the logistic map equation and we will get x_{200} decimal fractions.

Step 3 : Check condition.

Step 3a : If yes , then do 3 times the logistic map iteration and we will obtain the results are decimal fractions , such as x_{203}.

Step 3b : if not, so the encyption process is done for all part of image and we will obtain encrypted image.

Step 4 : Check whether the pixel block is the last or not.

Step 4a : If yes,Select the first 15 number behind the decimal from decimal fraction that has been placed before (for example x_{203}) , that are the result of 3 iterations logistic map. Then divide 15 number to 5 integer with each integernya has 3 points . Then take as much $mod\ (n.m\ ,5\ )$ integer . Do operation mod 256 to each integer , so we get a Chaos integer number $mod\ (n.m\ ,5\ )$ byte integer chaos . 1 byte chaos number is called key stream (KS) .

Step 4a.1 : Take the pixel value information at each grayscale as much as $mod\ (n.m, 5)$. Each 1 byte information of the image is called P.

Step 4a.2 : Do step 5 $mod(n.m, 5)$times.

Step 4.b : If not , then do transformation from real to integer, like in the step 4a, but take by 5 integer. Then take 5 integer. Do Operation mod 256 to each integer, so we get 5 bytes chaos integer number.

Step 4b.1 : Take the pixel information at each pixel grayscale by 5 . Each 1 byte information of the image is called P.

Step 4b.2 : Do the step 5 by 5 times.

Step 5 : Do bitwiseXOR operation on each byte *chaos integer number*withevery byte image data. Otherwise, do: $KS(j)\ \oplus P(j)$

Step 6 : Back to Step 3.

Information so that the original image data can be accessed again, it must be done the decryption process. Than the decryption process is the reverse process of encryption. Then, input the decryption process is an image that has been encrypted and the result is the original image. Key used in the decryption process must also be the same as the key used during the encryption process.

2.2 Results

Tests performed to see the computational time and the resistance of chaos-based encryption algorithm against brute force attack and known plaintext attack. Key space analysis and key sensitivity analysis were done to the resistance againts brute force atttack, while analysis of uniform distribution of pixel values (histogram analysis) were done to the resistance againts known plaintext attack.

A. Analysis of Encryption Time

Tests toward all digital image test data, performed using the same key value for both encryption and decryption process. Parameter values that used are $x_0\ =$ 0.1 and= 4 . The test results of the cat grayscale digital image (1-5) and lena (6-

10) with variety of sizes shown in Table 2.1 :

Table 2.1

Test Data	Pixel Size	Image Name	Average Encryption Time (second)	Average Decryption time (Second)
1.	2560 x 1920	Cat	99.8870000839	98.3209999648
2.	1280 x 960		25.0730001926	23.10300073012
3.	640 x 480		6.14599990845	7.44099995434
4.	320 x 240		1.46299982071	1.48399996758
5.	80 x 60		0.0940001010895	0.109000205994
6.	2048 x 2048	Lena	87.3000001907	88.1790001392
7.	1024 x 1024		21.6100001335	21.6159999371
8.	512 x 512		5.30499982834	5.35699987411
9.	256 x 256		1.29999995232	1.27800011635
10.	128 x 128		0.353000164032	0.371999979019

While the test results of the cat colored digital image (1-5) and lena (6-10) with variety of sizes shown in Table 2.2 :

Table2.2

Test Data	Pixel Size	Image Name	Average Encryption Time (second)	Average Decryption time (Second)
1.	2560 x 1920	Cat	182.141000032	183.129999923
2.	1280 x 960		45.3589999676	43.8740000725
3.	640 x 480		11.4220001698	11.0429999828
4.	320 x 240		3.10699987411	2.74099993706
5.	80 x 60		0.138999938965	0.19200000876
6.	2048 x 2048	Lena	154.983999968	156.345999956
7.	1024 x 1024		38.5360000134	38.9079999924
8.	512 x 512		9.66899991035	9.90400004387
9.	256 x 256		2.45600008965	2.59000000954
10.	128 x 128		0.605999946594	0.79069999925

B. Key Space Analysis

The random number generator which was used to generate key stream is logistic map. Keys that are used on logistic map are x_0 and λ, where x_0 and λ are real number. If we use a higher level of precision, for example 64-bit double precision IEEE standard, the precision level will reach 10^{-15}. So, the total possibilities of keys are $10^{15}x10^{15} = 10^{30}$.

C. Key Sensitivity Analysis

The value of the key that is used is always same for each digital image test data in this paper . While the decryption process will be tested with many differentkey value. The results are presented in Tables 2.3, 2.4 and 2.5 .

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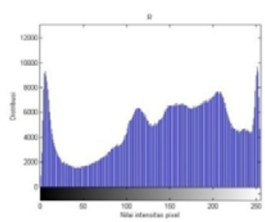
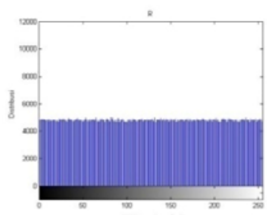
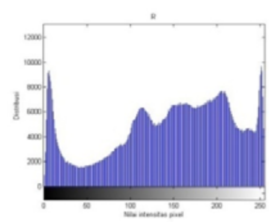
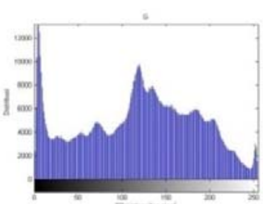
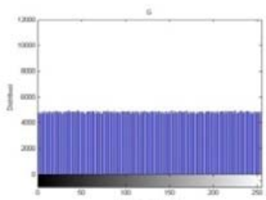
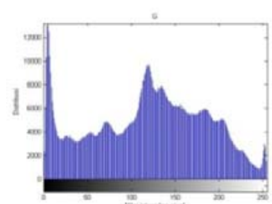
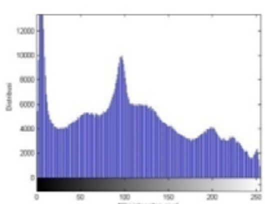
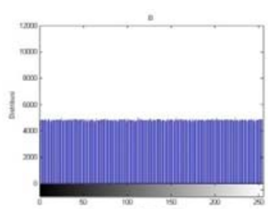
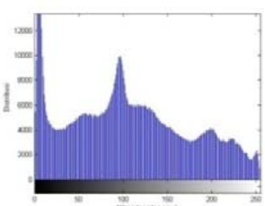
Table 2.3 The result of key sensitivity (case-1)

Oginal Image	**Encryption Result ($x_0 = 0.1$, $\lambda = 4$)**	**Decryption Result ($x_0 = 0.1$, $\lambda = 4$)**
		
		
		
		
	Same x_0	

In Table 4.3 are shown the results of the encryption and decryption process simulation using catimage with the same key that is $x_0 = 0.1$, $\lambda = 4$. Thus seen that the decryption process succeeded in opening the original data . Histogram display for each column in a row that the components R , G , and B shows the distribution of pixel values . Look at the histogram of original image and decrypt result image are very similar and have been proved also with the help of python that its values are the same. Table 4.4 shows the decryption attempt to use a different key to the encryption key.

Table 2.4 The result of key sensitivity (case-2)

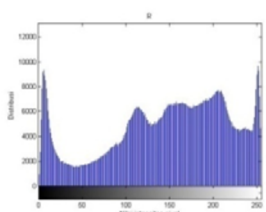
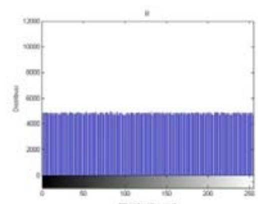
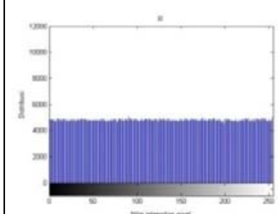
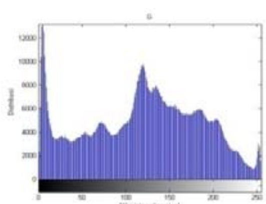
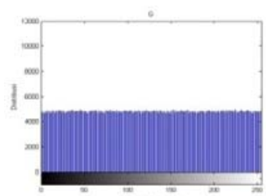
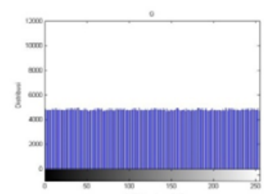
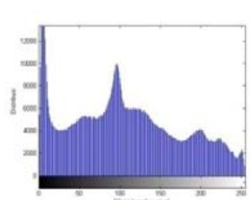
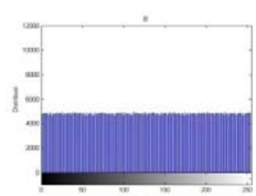
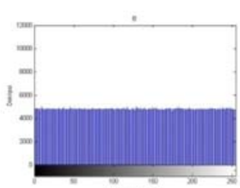
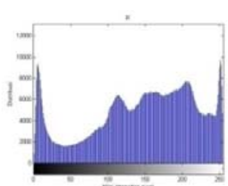
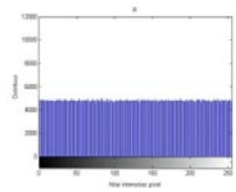
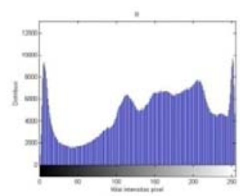
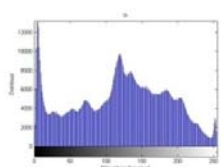
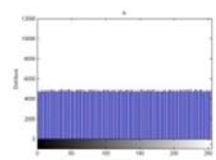
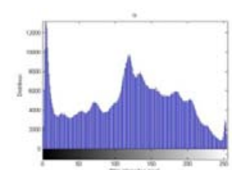
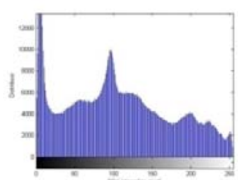
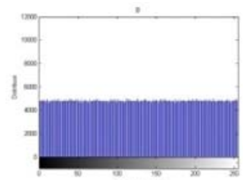
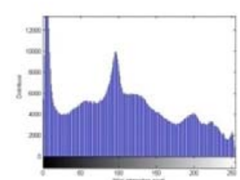
Original Image	**Encryption Result ($x_0 = 0.1$, $\lambda = 4$)**	**Decryption Result ($x_0 = 0.1000000000000001$, $\lambda = 4$)**
		
		
		
		
	Difference of x_0 Valueis10^{-16}	

Table 2.5 The result of key sensitivity (case-3)

Original Image	**Encryption Result** **($x_0 = 0.1$, $\lambda = 4$)**	**Decryption Result** **($x_0 = 0.10000000000000001$, $\lambda = 4$)**
		
		
		
		
	Difference of x_0 Valueis10^{-17}	

The test has tested with difference value ofx_0 when the decryption proccestox_0 when the encryption proccess.Shown in Table 4.4 that the attempt to decrypt using a key difference between the value x_0 by 10^{-16} did not succed to get the original image . But when the difference reaches $10^{-17}$the decryption proccess got the information of original image . It shows that the numbers 0.1 and 0.10000000000000001 is considered to be the same number that is 0.1. So we get the sensitivity of this algorithm is up to 10^{-16} .

3. Concluding Remarks

This algorithm is so hard to be cracked by brute force attack. Because the key space reach to 10^{30}, so it needs long time to find the key. Beside that, there are high key sensitivity factor that up to 10^{-16} and also because of random unpattern key streamfactor makes this algorithm is hard to be cracked to get the original data.

References

[1] Kocarev, L., & Lian, S, 2011, Chaos-based cyrptography. Berlin Heidelberg : Springer-Verlag.

[2] Pareek, N.K., Patidar, V., Sud, K.K., 2006, Image encryption using chaotic logistic map. *Journal of Image and Vision Computing,* 24, 926-934.

[3] Patidar, V., Pareek, N.K., Sud, K.K., 2009, A new subtitution-diffusion based image cipher using chaotic standard and logistic maps.*Journal of Communications in Nonlinear Science and Numerical Simulation*, 14, 3056-3075.

[4] Stallings, W, 2011,*Computer and Network Security : Principle and Practice* (5 ed.). New York: Prentice hall.

[5] Suryadi MT, 2013, New Chaotic Algorithm for Video Encryption, 4th The International Symposium on Chaos Revolution in Science, Technology and Society 2013, Jakarta, August, 28-29.

[6] Zhang, W., Wong, K.W., Yu, H., Zhu, Z.L, 2013, An image encryption scheme using reverse 2-dimensional chaotic map and dependent diffusion. *Journal of Communications in Nonlinear Science and Numerical Simulation*, 18, 2066-2080.

APPLICATION OFIF-THEN MULTI SOFT SET

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Abstract. A soft set as a relatively new mathematical tool for dealing with uncertainties was first introduced by Molodtsovhas experiencedrapid growth. Variousapplicationsof softsetfor the purpose ofdecision-makinghave beenshown by several researchers. From various studies presen tedmostly shows the role of soft sets as a tool in the collection o fthe various attributes needed by a person to determine which decisions will be taken. The development ofthe use of soft setmay actually be more than that, this paper will show how soft set can play a rolein the decisionmade bya personbased onahistory ofdecisionsthathave been madebysomepeopleearlierandused asa referenceforthe nextdecision. Therefore, this paper introduce an (if-then) multi soft sets as a developments of application of soft set which is stated in the form if (antecedent) then (consequence) with antecedent and consequence are derived from previously several decisions that have been made by people when using a soft set as a tool to help them for making a decision.

Key words and Phrases: Soft Set, Multi Soft Set, *If – then*, Decision

1. Introduction

The buyer-oriented industry puts the buyer at the highlight of everything business does. The ability of industrial management to understand the buyer could make a significant increase in its sales. Understanding the buyer or any kind of terms that refer to deep recognizing the buyer such as buyer persona could help the industry understand what the buyer really want. Even though many people who do the marketing knowing the big impact of understanding the buyer but very few companies do a research of buyer (even a simple research), they just regarding this problem as an sense or intuition that the buyer engage with their business product's. Most corporations expect that collecting the information/insights about the buyer just wasting time effort that make an ineffective work and spending a lot of money that will lessen their sales.

Today, people are very stingy to give their personal information even when the salespersons do the very warm welcome and humble conversation. It is difficult to satisfying in the field of buyer persona to obtain reference of the buyer such as buyer demographics, buyer's role, buyer psychology or deeper personal information about the buyer. This could be understood that buyer keep their individual information for security reason. Therefore, salesperson should be exerting all effort to observe the buyer even though only using a simple observation. Buyer just asking what they need and they do not want to be asked outside of the things they asked. Buyers get annoyed when the shop assistant started asking personal things. They will only provide personal information that they feel is important to be known by the shop owner, such as their address in which the company must deliver the goods they had bought.

Once the company has gathered a sufficient amount of simple but deep observation then all company's team participate inevaluating the buyers. Rough observation of buyers could be put in the form of soft set. Discussion should be made as an effort to approximate accurately reflect who the buyer is and their decision that have been made that could be made as a basis for recommendation to the next buyer in choosing company's product.

In this article will show the simple steps of marketing and sales based on soft set theory to quickly understand the buyer via simple observation of the buyer.Decision that has been made by buyer to purchase is a result of complicated policy from point of view of buyer. Choosing one product to be purchased could be started by describing the product they want using some simple characteristics, attributes, information or knowledge they have about those product. Any parameters which had been regarded as an important characteristic that might be owned by the product to be bought could be collected in the structure of mathematical notion. Collection of those parameters can be laid on the form of soft set theory. Soft set theory first introduced by Molodtsov [15] in 1999 and has been applied in many fields by researchers. Hakim et al. [7] had proposed a recommendation analysis as a buyer tool to assist their decision in purchasing a product. Many researchers (Chen et al. [2], Feng et al. [5][6], Herawan and Mat Deris [8], Jiang et al. [9], Kong et al. [11], Maji et al. [13][14], Roy and Maji [18]) mostly show the role ofsoftsetsas a tool inthe collection ofvariousattributes or parameters of objects neededbyapersonand thendetermineusing some calculations whichdecisionswill betaken. The development ofthe use of softsetmay actually be morethanthat, this paperwill showhowsoftsetcanplay a rolein the decisionmade bya personbased onahistory ofdecisionsthathave been made by some people earlier and used asa reference for the nextdecision.

Deciding a product to be purchased is a difficult matter for a buyer. Hakim et al. [7] has introduced a recommendation system based on soft set theory to purchase a product from buyer side. This recommendation analysis is an advantage for buyers in helping them to determine the product they need. This paper also trying to use a soft set theory from a view of store team to observe the personality of buyer by means of the ability of the store owner and store assistant to evaluate their buyer when purchase goods. Since store team observation also will be put in the form of soft set theorythen we have a structure of dual usage of soft set theory. First is recommendation analysis for buyer in choosing a store's product and secondly to assist buyer in finding out the products based on observations of the store team to its buyers. Both of them are based on the parameters which are collected from the information of buyer and store team. Therefore, it can be seen as two parts, which are condition and decision model problem. The 'condition' and the 'decision' parts will be determined after remarking two usage of soft set which first useis application of soft set in buyer's view when purchasing product and second use is application of soft set in store team observation to their customers.

We will develop this new application of soft set in the form *if* (antecedent) *then* (consequence) with antecedent as a condition attribute and consequence as a decision attribute that are derived from previously several decisions which had been made by other buyers. Because it involves a condition and decision attribute, we need a language of 'decision rules'. A decision rule is an implication in the form *if* A *then* B, where A is called the 'condition' and B the 'decision' of the rule. Decision rules state relationship between conditions and decisions. In this paper, we are trying to combine the decision rules and dual soft sets that will produce a new application of soft set which can be known as *if-then* multi soft-set. This application not only helping buyer in deciding the product to be chosen but also help the store to map their buyer when determining the product needed.

2. Literature Reviews and Main Results

2.1. Soft Set Theory

Molodtsov [15] first defined a soft set which is a family of objects whose definitions depend on a set of parameter. Let U be an initial universe of objects, E be the set of adequate parameters in relation to objects in U. Adequate parametrization is desired to avoid some difficulties when using probability theory, fuzzy sets theory and interval mathematics which are in common used as mathematical tool for dealing with uncertainties. The definition of soft set is given as follows.

Definition 2.1. (Molodtsov [15]).*A pair (F, E) is called a soft set over U if and only if F is a mapping of E into the set of all subsets of the set U.*

From definition, a soft set (F, E) over the universe$U$ is a parameterized family that gives an approximate description of the objects in$U$.Let e any parameter in E, $e \in E$, the subset$F(e) \subseteq U$may be considered as the set of$e$-approximate elements in the soft set (F, E).

Example 2.1. *Let us consider a soft set (F, E)which describes the "attractiveness of houses" that Mr. X is considering to purchase.*

U – is the set of houses under Mr. X consideration

E – is the set of parameters. Each parameter is a word or a sentence

E = {expensive, beatiful, wooden, cheap, in the green surroundings, modern, in good repair, in bad repair}

In this example, to define a soft set means to point out expensive houses, that shows which houses are expensive due to the dominating parameter is 'expensive' compared to other parameters that are possessed by the house, in the green surrounding houses, which shows houses that their surrounding are greener than other, and so on. Molodtsov [15] also stated that soft set theory has an opposite approach which is usually done in classical mathematics that should construct a mathematical model of an object and define the notion of the exact solution of this model. Soft set theory uses an approximate nature as an initial description of the objects under consideration and does not need to define the notion of exact solution. Common mathematical tools to solve complicated decision problems with uncertainties are probability theory, fuzzy theory and interval mathematics, but there will get difficulties on using them. Probability theory must perform a large number of trials, fuzzy theory must set the membership function in each particular case and the nature of the membership function is extremely individual, interval mathematics should construct an interval estimate for exact solution of a problem but not sufficiently adaptable for problem with different uncertainties. To avoid these difficulties, in the soft set theory, when someone is faced to the decision problems with many uncertainties, problem or object is determined by the ability of the person to explain various things related to that object. Various things that might be related are referred to as the object parameter in the soft set. He or she could express the parameters use any information that might be possessed by the objects. This relevant information refers to the necessary parameter of the objects. The necessary parameters could be a particular interest that he or she can express their preference, knowledge, perception or common words in a simple way to the objects under consideration. Parameters attached to the objects are said to indispensable if he or she considers that the information involved to identifying a problem is sufficient to elucidate the objects.Since object parameters have been determined and then give a fair valuation to each object based on those parameterwould yield a solution that gives a suggestion to make a decision. Setting the objects and their necessary information using words and sentences, real numbers, function, mappings, etc., is a parametrization process that makes soft set theory applicable in practice.

Maji et al. [13] has extended example 2.1 to decision making problem of choosing six houses based on the attractiveness of houses as a houses parameters. Some parameters are absolutely belonging to some houses and some parameters are absolutely not belonging by some houses. Their example of choosing of houses problem based on soft set has initiated many important applied and theoretical researches that have been achieved in soft set decision making problem. However, soft set theory has not been yet find out the right format to the solution of soft set theory due to many research using binary, fuzzy membership or interval valued for parameters of objects valuation which actually should have been avoided as notified by Molodtsov [15].

2.2. Soft Solution of Soft Set Theory.

Maji et al.[13][14] applied the theory of soft set to solve a decision making problem using example which was described by Molodtsov. They used tabular representation with the entry is 1 if an object has a particular parameter, and zero if it does not has then decision is based on the maximum of cumulative number of objects that have parameters of the soft set.To handle the binary valuation, Maji et al. [12] tried to introduce the *W*-soft set or weighted soft sets, but their effortwould not give a new approach to decision analysis caused by the weightsare multiplied to each parameter (as attribute) and hence, this method would not change the result.Later, Herawan and Mat Deris [8] and Zou and Xiao [20] have proven that soft set could be transformed to binary information system. Maji et al. [13] also used rough set theory to reduce the parameters that have been hold to every object in the universe. Unfortunately, rather than optimize the worth of parameter as necessaryinformation; they preferred to reduce the parameter. The information involved in the parameter will be loosed. Molodtsov has insisted the adequacy of parameterization to objects of universe rather than reducing the parameter that has been belonged to every object. Due to binary value of the entries, their decision result gives an exact solution that might contradicted to the philosophy of the initial Molodtsov's soft set that insisted the approximation to the result which is caused by soft information accepted in a parametrization family of soft set.Chen et al.[2] and Kong et al.[11] also wanted to reduce the parameter of the objects, but

Molodtsov already pointed out that the expansion of the set of parameters may be useful due to the expansion of parameters will give more detailed description of the objects. Since the adequacies of parameter arecrucial in soft set theory to describe the houses, reduction of parameters may bring out a set of indispensable parameter from a set of parameters. This reduction parameter expected to deduct valuable information from a set of indispensable parameter of soft set.

Roy and Maji [18] combined the fuzzy set and soft set, fuzzy numbers is used to evaluate the value of the parameter's judgment for each object. This idea develops to the hybrid theory of fuzzy soft set. This fuzzy soft set also initiated by Yang et al. [19]. They also said that rather than using binary value, it will be better to use the degree of membership to represent the objects which hold the parameter. Since the parameter's value of each object filled by fuzzy numbers, there must be an expert to determine the membership values that represent the matching number for each house. It may become more difficult since the valuation of parameters of the objects are on the interval-valued fuzzy number (Feng et al. [5], Jiang et al. [9]). An expert should give not only matching number of parameter but should determine the lowest and the highest number as the value of the objects parameters. Molodtsov had been stated that this is the nature difficulties when dealing with fuzzy numbers and should be avoided. The idea to substitute the value of each object parameters using binary, fuzzy number or interval-valued fuzzy number may still be used as a reference because it gives results that can be used as a benchmark for a person in making decision.

All researchers on soft set in decision making could be grouped into two groups. First, researchers that treat the soft set as an attribute of information system (Herawan and Mat Deris [8], Zou and Xiao [20]) then using Rough Set to handle the vagueness for making a decision (Feng et al. [6]). Second, researchers that use fuzzy theory to soft set (Jun et al. [10], Feng et al. [5] and Jiang et al. [9]). Both of them gave techniques which produce best decision based on binary or fuzzy number rather than recommendation that may be little bit more satisfying Molodtsov's soft set philosophy. Of the entire study could be seen that the whole objects under consideration was assessed through the parameters by the decision makers and will get the solution in the form of a subset of the objects itself that each of them has a dominating parameters. According to this understanding we will give the definition for soft solution of soft sets.

From that definition 2.1., a soft set (F, E) over the universe$U$ is a parameterized family that gives an approximate description of the objects in$U$.Let e any parameter in E,$e \in E$, the subset$F(e) \subseteq U$ may be considered as the set of$e$-approximate elements in the soft set (F, E). It is worth noting that the sets $F(e)$ may be arbitrary. Some of them may be empty, some may have nonempty intersection. That is, the solution of the soft set is a set which are a subset of object and a subset of parameters that shows the objects and its parameters.

Definition 2.2. (soft solution). *A pair (F', E') over U' is said to be a soft solution of soft set (F, E) over U if and only if*

i) $U' \subseteq U$

ii) $\{e_{|U'} \mid e \in E\} = E'$ *where* $e_{|U'}$ *is the restriction parameter of e to U'*

iii) *F' is a mapping of E' into the set of all subsets of the set U'*

We shall use the notion of restriction parameter of $e \in E'$ to U' in order to obtain the parameters which dominate an object compared to other parameters that may be possessed by those objects.

A soft set (F, E) over U might be considered as an information system (U, AT) (Demri and Orlowska [3]) such that $AT = \{F\}$ and value of a mapping function of $F = e \in E$ make available the same information about objects from U. It is a common thing to identify a wide range of matters (parameters) relating to the object and then create a collection of objects that possess this parameters. To compose this intuition, for a given soft set $S = (F, E)$ over U, we define a soft set formal context $S = (U, E, F)$ where U and E are non-empty sets whose elements are interpreted as objects and parameters (features), respectively, and $F \subseteq U$ x E is a binary relation. If $x \in U$ and $e \in E$ and $(x, e) \in F$, then the object x is said to have the feature e.In this concept, the soft set formal context provide the following mappings *ext*: $\mathbf{P}(E) \rightarrow \mathbf{P}(U)$, that shows extensional information for objects under consideration. This means an object parameters may be able to be expanded on someone views as the set of those objects that possess the parameters.

2.3. Multi Soft Set

In the situation where a lot of things involved in decision-making, these situation can be separated and described in some soft sets. Here are some thought that underlying the application of multi soft sets.

Definition 2.3. *For all $X \subseteq U$ and $e \subseteq E$ we define $ext(E) \overset{def}{\Rightarrow} \{x \in U \mid (x,e) \in F$, for every $e \in E\}$;$ext(E)$ is referred to as the extent of E.*

A soft set formal context $S = (U, E, F)$ is a urnfor a collection of soft sets. Not necessarily soft set formal context will only give one soft set. $S = (U. E, F)$ could be viewed as Multi Soft Set, say dual soft set S_1 and S_2 where $S_1, S_2 \subseteq S$ and S_1 is soft set (F_1, E_1) over U_1, S_2 soft set (F_2, E_2) over U_2, and $U_1, U_2 \subseteq U$ and $E_1, E_2 \subseteq E$ and $E_1 \cap E_2 = \emptyset$

Lemma2.1. *For Soft set formal context $S = (U. E, F)$, $S_1, S_2 \subseteq S$ and S_1 is soft set (F_1, E_1) over U_1, S_2 soft set (F_2, E_2) over U_2, for all $U_1, U_2 \subseteq U$ and $E_1, E_2 \subseteq E$ if $E_1 \cap E_2 = \emptyset$, then $ext(E_1) \cap ext(E_2) = \emptyset$*

PROOF. Let E_1, $E_2 \subseteq E$. Assume that $x \in U_1 \subseteq U$ and for every $e \in E_1 \subseteq E$then $ext(E_1) \overset{def}{\Rightarrow} \{x \in U_1 \mid (x, e) \in F_1$, for every $e \in E_1 \subseteq E\}$ since $E_1, E_2 \subseteq E$and $E_1 \cap E_2 = \emptyset$,$x \in U_1 \subseteq U$ and for every $e \in E_1 \subseteq E$then $(x,e) \in F_1$ that is $ext(E_1)$ and never $ext(E_2)$.□

Lemma 2.1 shows that a soft formal context can be divided into a number of soft sets (Multi Soft Sets) with each object and its parameters are different but still in the same context. It is in line with Alkhazaleh and Salleh [1] that has introduced the definition of Fuzzy Soft Multiset with a collection of universes and reminds that any change in the order of universes will produce a different Fuzzy Soft Multiset. But they did not explicitly state the relationship between each soft set inside Multi Soft Set with a different multiset. This notion can be exemplified as follows, in a furniture store, the buyer establish a soft set to choose which one to buy based on desired parameters, while the shop owner build a soft set to observe the behavior of the buyer to make his choice. There will be relation between furniture chosen with the buyer performance. In this paper we will use the definition of Multi Soft Set in the formal context and clearly state the relationship between soft sets with *if-then* decision rules.

3. The Proposes Application Of *if-then* Multi Soft Sets.

In this paper we will develop again the examples given by Hakim et al [7]

which illustrate a user interface of soft set recommendation analysis for purchasing furniture products in some furniture store. System (figure 3.2) displays three columns, first columns consists of customer identification and buyers are offered to get assistant from furniture expert for choosing and question of some specific purpose in intending buying the furniture. All collections of furniture items are shown in second column and buyer was asked to choose one of collections. In this example, buyers choose dining chairs then the third column display all collection of dining chairs. Four selected chairs (figure 3.3) are chosen by customer and buyers could determine their own requirements for their dining chairs. Buyer has several things that he thought as a dining chairs precondition, he could type any perspective inside the form, for example, 'match with my dining room decoration', 'fit the space of my dining room', 'cheap', 'comfort', 'classic' and 'wood color'. This could be expressed in the form of soft set. A soft set (F_1, E_1) of this example could be described as the preconditions of the chairs which buyer is going to buy.
U_1 – is the set of chairs under consideration {CH1, CH2, CH3, CH4}
E_1– is the set of parameters. Each parameter is a word or a sentence.
E_1= {match with my dining room decoration, fit the space of my dining room, cheap, comfort, classic and wood color}

Those preconditions could be regarded as parameters of each chair. He thought that, those information/ knowledge/parameters are necessary parameters for him to choose a chair that he need for inviting a special guest for dinner. Soft set has applied here, that someone could use any parametrization he wants for purchasing chairs. It might he only knows what he need and conditions that he must consider putting the chairs then. Meanwhile, in this cases we offer a judgment from expert based on buyer's precondition which is available in the form below the 'customer request', this form shows what Expert Says with the valuation of each chair below of its pictures. In the second column, it also displays the form of customer evaluation that he could determine his own judgment for each chair. The simple act to evaluate the selected items is to compare them in a fairly flexible way by giving a mark to the chairs that meet his requirements. More asterisks more meet parameters (table 3.1).

Table 3.1. Evaluation of chairs' parameter by buyer

Chair	**Match**	**Fit the space**	**cheap**	**Comfort**	**Classic**	**Wood color**
Chair1	***	**	*	**	**	***
Chair2	*	*	*	***	***	***
Chair3	**	**	***	*	**	*
Chair4	*	**	**	*	*	*

To better utilize information from the tables and provide added value to our recommendation for Mr. X, we will use multidimensional scaling techniques. Non-metric multidimensional scaling techniques are common techniques which based on ordinal or qualitative rankings of similarities data (Kruskal [12]). Therefore, table 3.1 needs to be transformed via the numbers into an ordinal table.Using the software R (R Development Core Team [17]) with*vegan*package and *metaMDS*procedure (Dixon and Palmer [4]), we get the mapping of houses and its parameters ranking on figure 3.1.

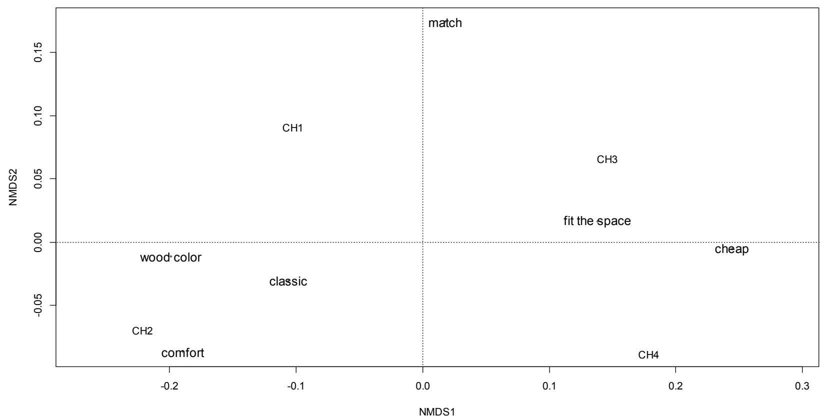

Figure 3.1. Multidimensional scaling plot of chairs and its parameters

Based on the figure 3.1 the soft solution of soft set for this problem is

Soft solution (F_1',E_1')	=	{(Match the dining room decoration) = CH1, (Woodcolor, Comfort, Classic) = CH2, (Fit the space of dining room, Cheap, Match the dining room decoration) = CH3, (Cheap) = CH4}

This set of soft solution is used as a recommendation forbuyer to purchase the chairs. This soft solution is a result of soft set using hierarchical clustering and multidimensional scaling techniques. Outcome of this solution is a recommendation based on buyer's evaluation, for example, the first picture show that the chair tends to match with the dining room decoration while second picture meet a lot of customer requests, i.e., wood color, comfort and classic style. The final decision is verified by customer to buy that chair. The last row could be utilized as an offer to buyer for buying another product which is usually bought by others while buying that chair.

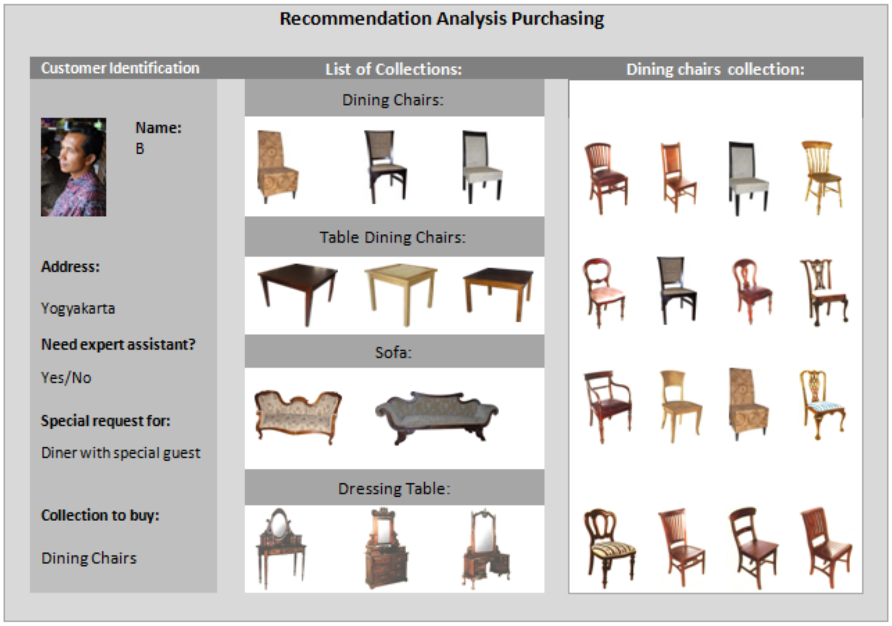

Figure 3.2. Soft set recommendation analysis first page.

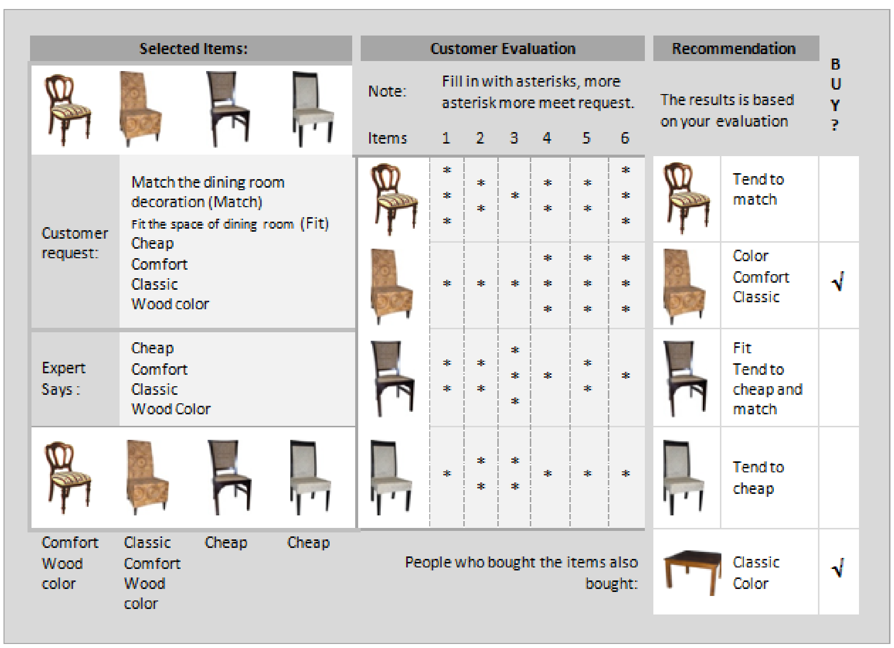

Figure 3.3. Soft set recommendation analysis second page.

Nowadays buyers are miserly to give personal information due to security reason and get annoyed when shop assistant started asking personal things. Shop assistant also cannot force the buyers ask for personal information, but much better if observing the behavior of buyers when selecting products and make their choice. Simple research will be carried out if the shop owner does not assume the arrival of buyer to their shop only as a destiny. Buyer that has already coming to their shops should be noticed use any kind of characteristics or parameters which the owner or shop assistant could do. Some simple parameters that might could be used to differentiating one person to another such as, appearance of buyers, style of buyers when asking something, gesture of buyers, speaking style of buyer and so on.

In this example, the furniture store owner and their team trying to observe their buyer using several parameter which could be put in the form of set {tidy appearance, looks wealthy, age-old, too much questions, modern lifestyle, complicated requests, busy and in hurry, too much bargain} that they think sufficient to evaluate their buyer. Other owners could add or reduce the parameter used in this set, depend on the observation to their own buyers. A soft set (F_2, E_2) of this example could be described as the behavior of buyers as the result of observation of the owner

U_2 – is the set of buyers under consideration {A, B, C, D}
E_2– is the set of parameters. Each parameter is a word or a sentence.
E_2= {{tidy appearance, looks wealthy, age-old, too much questions, modern lifestyle, complicated requests, busy and in hurry, too much bargain}.

The owners could ask his/her employee to help him in judging their buyers. Of course, the owner and other employee does not need to become an expert in advance at the field of human personality evaluation. They just give simple evalution in accordance to their ability to observe and what they feel about their buyer. In this example suppose the owner choose 4 buyer that had been made a transaction with him. Those buyer will be evaluated based on observation in which the owner and its team remembering again behavior of the buyer when they was in their store. The way of dealing with evaluation usually using ranking or rating to the objects under consideration and express their perception in an easiest and fairly flexible way. The simplest expression is give ranking by using an asterisk that show more asteriks in the parameter means more adjacent the parameter belonging to the object. The owner will give more asteriks if one buyer meet the parameters than other buyer. For instance, in parameter 'tidy appearance' Mr. B seems the most tidy than others. Mr. C seem as tidy as Mr. D, even though both of them not really equal tidy in appearance. Mr. A is the most not tidy compared to other three buyers. Mr. C looks the wealthiest than others, Mr. A and Mr. D looks same wealthy and Mr. B looks not so wealthy. Evaluation could be continued to the next parameters. Tabular representation of the buyers and parameters would be useful to describe the response of shop owner and his team in evaluating their buyers (table 3.2).

Table 3.2. Evaluation of shop owner to his/her buyers.

	Tidy appearance	**Looks wealthy**	**Age-old**	**Too much question**	**Modern lifestyle**	**Complicated requests**	**Busy and in hurry**	**Too much bargain**
A	*	**	***	**	*	**	*	**
B	***	*	***	***	*	***	*	***
C	**	***	**	*	***	*	***	**
D	**	**	*	***	***	*	**	*

Doing the same thing to the first soft set, after evaluation and bring table 3.2 to numbers and used multidimensional scaling technique to mapping those table, we get the soft solution for the second soft set. Soft solution of soft set for the behavior of buyers is

Soft solution $(F_2{'},E_2{'})$ = {(Age-old, Too much bargain) = A,
(Tidy appearance, Too much questions,
Complicated requests) = B,
(Wealthy, Busy and hurry) = C,
(Modern lifestyle) = D}

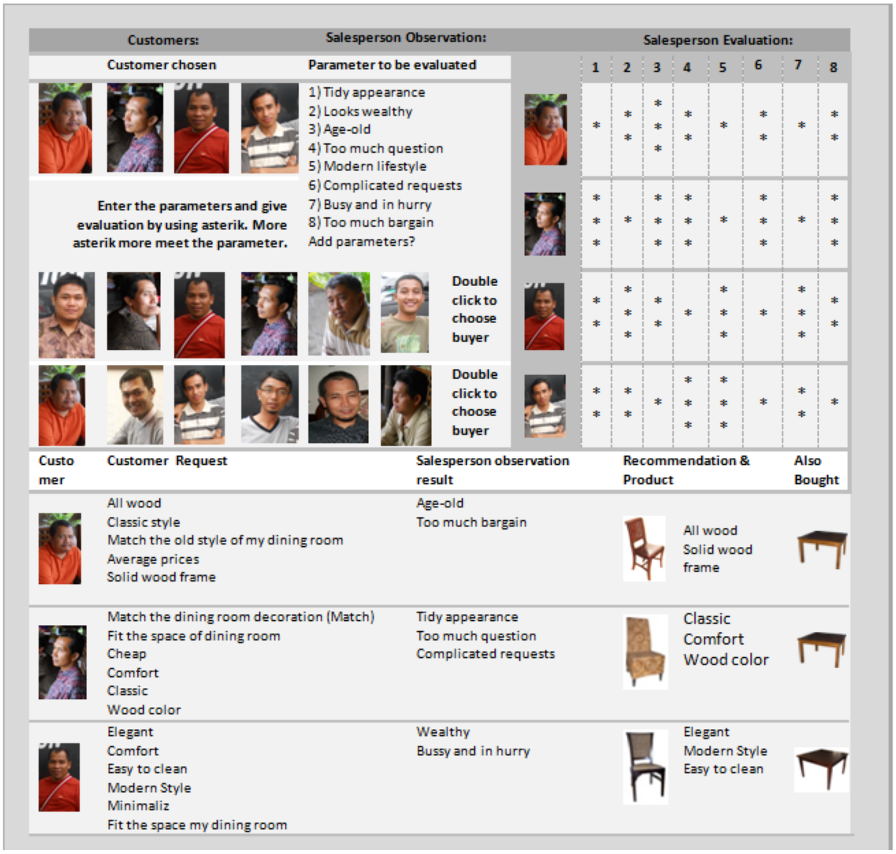

Figure 3.4. Interface of application *if-then* multi soft set

From this two soft set (F_1, E_1) and (F_2, E_2) give a result of two soft solution which are $(F_1{'}, E_1{'})$ and $(F_2{'}, E_2{'})$ and due to high relationship between two soft set, the owner could get the decision rules of two soft solution which are

If $(F_1{'}, E_1{'})$ *then* $(F_2{'}, E_2{'})$ or *if* $(F_2{'}, E_2{'})$ *then* $(F_1{'}, E_1{'})$

Say, the owner would like to take one of those buyer to see the decision rules of multi soft set, say Mr. B, then he will get the rules,

if	Mr. B bought chair {Classic, Comfort, Wood color} = (Ch2)	*then*	Mr. B is {Tidy appearance, Too much questions, Complicated requests}

or

if	Mr. B is {Tidy appearance, Too much questions, Complicated requests}	*then*	Mr. B bought chair {Classic, Comfort, Wood color} = (Ch2)

Second rule seems reasonable for recommendation which will be used by the owner and his sales person to a buyer who have behavior looks like Mr. B. Of course this rule will not be disclosed to the buyer, because the rules are based on the assessment of shopkeeper to their buyers quietly. Even though this recommendation is not exact decision but this rules could help the owner and the sales person to assist the buyers while they are determining to choose one chair from several chairs of dining room. Figure3.4. shows the interface of recommendation analysis of the owner to their buyers. This work has already shown the applied of soft set when it is implemented in the rules *if-then*. The usage of Multi Soft Set, could help not only buyer when he/she need to choose the object he wants but also help the shop owner to give recommendation to his/her buyer based on buyer behavior.

4. Conclusion

Soft set as a new mathematical tool in decision making already give lack of restrictions to the one who using them to get the final decision. Anyone could use any parameters in deciding which objects will be choosen. Frommany previous studies mostly shows the role ofsoftsetsas a tool inthe collection ofthe variousattributes neededbyapersontodeterminewhichdecisionswill betaken however in this paper we have already shown the development ofthe use of softsetto multi soft set and its lemma.We show how*if-then* multi softsetcanplay a rolein the decisionmade bya personbased onahistory ofdecisionsthathave been madebysomepeopleearlierandused asa referenceforthe nextdecision.

References

[1] Alkhazaleh, S. and Salleh, A. R., Fuzzy Soft Multiset Theory, *Abstract and Applied Analysis*, vol. 2012, Article ID 350603, 20 pages, 2012. doi:10.1155/2012/350603

[2] Chen, D., Tsang, E.C.C., Yeung, D.S., and Wang, X., 2005, The Parameterization Reduction of Soft Sets and its Applications, *Computers and Mathematics with Applications* 49 (2005) 757-763.

[3] Demri, S. P. and Orlowska, E. S., 2002, *Incomplete Information: Structure, Inference, Complexity*, Springer-Verlag, Berlin Heidelberg, New York.

[4] Dixon, P., and Palmer, M.W., 2003, Vegan, a package of R function for community ecology, *Journal of Vegetation Science* 14, 927-930.

[5] Feng, F., Jun, Y.B., Liu, X. and Li, L., 2010, An adjustable approach to fuzzy soft set based decisionmaking. *Journal of Computational and Applied Mathematics* 234, 10–20.

[6] Feng, F, Liu, X., Leoreanu-Fotea, V., and Jun, Y. B., 2011, Soft set and soft rough sets, *Information Sciences* 181: 1125 – 1137

[7] Hakim, RBF., Subanar, Winarko, E., Recommendation Analysis Based on Soft Set for Purchasing Products., pp 831 – 848, *Proceedings of The 6th Southeast Asian Mathematical Society (SEAMS) International Conference of Mathematics and Its Application* 2011, pp 831 – 848.

[8] Herawan, T. and Mat Deris, M., 2011, A soft set approach for association rules mining, *Knowledge Based System* 24: 186 – 195

[9] Jiang, Y., Tang, Y. and Chen, Q., 2011, An adjustable approach to intuitionistic fuzzy soft sets based decision making, *Applied Mathematical Modelling* 35: 824-836

[10] Jun, Y. B., Lee, K. J., and Park, C. H., 2010, Fuzzy soft sets theory applied to BCK/BCI-algebras, *Computers and Mathematics with Applications* 59: 3180-3192

[11] Kong, Z., Gao, L., Wang, L., and Li, S., The normal parameter reduction of soft sets and itsalgorithm, *Comput. Math. Appl*., 56 (2008), 3029-3037.

[12] Kruskal, J. B., 1964, Nonmetric multidimensional scaling: A numerical method, *Psychometrika* 29, (1964), pp. 115-29.

[13] Maji, P.K., Roy, A. R., and Biswas, R., 2002, An Application of Soft Sets in A Decision Making Problem, *Computers and Mathematics with Applications* 44: 1077-1083.

[14] Maji, P.K., Roy, A. R., and Biswas, R., 2003, Soft Sets Theory, *Computers and Mathematics with Applications* 45: 555-562.

[15] Molodtsov, D., 1999, Soft Set Theory – First Results, *Computers and Mathematics with Applications* 37: 19 – 31.

[16] Nijkamp, P. and Soffer, A., 1979, Soft Multicriteria Decision Models for Urban Renewal Plans, *Researchmemorandum* no. 1979-5, Paper Sistemi Urbani, Torino.

[17] R Development Core Team, 2006, *R: A language and environment for statistical computing*, R Foundation for Statistical Computing, Vienna, Austria. (http://www.r-project.org/)

[18] Roy, A.R. and Maji, P.K., 2007, A Fuzzy Soft Set Theoretic Approach to Decision Making Problems, *Computational and Applied Mathematics* 203: 412-418

[19] Yang, X., Yu. D., Yang, J., and Wu, C., 2007, Generalization of soft set theory: From crisp to fuzzy case, in *Fuzzy Information and Engineering* (ICFIE), ASC 40: 345-354

[20] Zou, Y., and Xiao, Z., 2008, Data Analysis Approaches of Soft Sets under Incomplete Information, *Knowledge Based System* 21: 941 -945,

ITERATIVE UPWIND FINITE DIFFERENCE METHOD WITH COMPLETED RICHARDSON EXTRAPOLATION FOR STATE-CONSTRAINED OPTIMAL CONTROL PROBLEM

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Abstract. This paper proposes a numerical algorithm for solving state-constrained optimal feedback control problems. In this work, we use the Upwind Finite Difference Method introduced in Wang [24] in an iterative fashion to improve the speed of the method. In addition, with the combination of Completed Richardson Extrapolation (see Richards [15]) the accuracy of the method can be increased.

Key words and Phrases: Optimal Control, Hamilton-Jacobi-Bellman Equation, Finite Difference Method, Penalty Function, Richardson Extrapolation.

1. Introduction

In this article we present a numerical method for solving state-constrained optimal feedback control problems. Consider the following optimal problem,

$$\min_{u(t)\in U} \quad J(u) = \int_s^{t_f} L(x(t), u(t), t)\, dt + \phi(x(t_f))$$

$$\text{subject to} \quad \dot{x} = f(x(t), u(t), t), \qquad \text{for all } t \in (s, t_f],$$

$$x(s) = y,$$

where $x \in R^n$ and $u \in U \subset R^m$ are the state and control, $t_f > 0$ is a constant, $(s,y) \in [0,t_f) \times R^n$, , $y \in R^n$ is a given point. $L : R^{n+m+1} \to R$, $\varphi : R^n \to R$ and $f : R^{n+m+1} \to R^n$ are known functions.

If $s = 0$ and $y = x_0$ is fixed then the formulation is reduced to an open-loop optimal control problem. The solution to the open-loop problem constitutes an optimal control u along the optimal trajectory x from the initial state x_0. Unluckily, if perturbations exist in the state x such that the state is off the optimal trajectory then the corresponding optimal control solution is no longer available. On the other hand, the solution to the closed-loop problem is robust or stable because it is defined over a time-space region that contains the optimal trajectory. Hence, the corresponding optimal control still can be found if the disturbances are present in the state x.

For that purpose, we firstly need to translate the problem into first order partial differential equation called the Hamilton-Jacobi-Bellman(HJB) equation.

By defining the value function $V(s,y) = \inf J(s,y,u)\ u$

and using the Dynamic Programming approach, this problem can be converted to HJB,

$$\frac{\partial V}{\partial t} + \inf_u(\nabla V \cdot f(x,u,t) + L(x,u,t)) = 0 \tag{1}$$

with terminal condition

$$V(t_f,x) = \varphi(x(t_f)).$$

In this equation there are two unknowns, the value function V and the optimal control u.

Generally, the solution to HJB equation is continuous but nonsmooth. To deal with this nonsmooth solution, the concept of viscosity solution was introduced by P.L. Lions etc.[3], [4] and [10]. In the presence of constraints, H.M. Soner [18] broadened the definition of the viscosity solution to the constrained viscosity solution. More detail about viscosity solutions of HJB equation can be found in Bardi [1] and Fleming [6].

Although HJB equation theoretically has been solved, unfortunately, it is in general difficult to find the analytic solution. Hence, in practice numerical approximation is necessary. There are many numerical methods available in the literature for solving unconstrained HJB equation, such as in Bardi [1], Guo [7], [8], Huang [9], Wang [24], [25] to name but a few.

Among these methods, we are interested in the Upwind Finite Difference Method introduced in Wang [24] due to its simplicity. For one dimensional problems, the discretization of equation (1) is as follows:

$$\frac{V_i^{k+1} - V_i^k}{\Delta t} + \frac{1 + \mathrm{sign} f_i^k}{2} f_i^k \frac{V_{i+1}^k - V_i^k}{\Delta x} + \frac{1 - \mathrm{sign} f_i^k}{2} f_i^k \frac{V_{i-1}^k - V_i^k}{\Delta x} + L_i^k = 0$$

$$u_i^{k+1} = \arg\inf_u (f(x_i,u,t_{k+1})\frac{V_i^{k+1} - V_{i-1}^{k+1}}{\Delta x} + L(x_i,u,t_{k+1})$$

for $k = 1,...,N$ and $i = k,...,M - k$, where M,N are the number of partitions of spatial and time intervals respectively, $\mathrm{sign} f_i^k$ denotes the sign of f_i^k and

$$f_i^k = f(x_i,t_k,u_i^k), L_i^k = L(x_i,t_k,u_i^k), V_i^k \approx V(x_i,t_k), u_i^k \approx u(x_i,t_k).$$

This method is the Upwind Finite-Difference because it takes into account upwind directions from which characteristic information propagates. If $f_i^k > 0$ the scheme switches to the forward-difference scheme and to backward-difference scheme for the opposite sign.

However, there is a drawback, namely trapezoidal propagation of the spatial domain with each time step. This propagation causes a large initial region so that it leads to expensive computation due to a greater number of computed grid points. To address this problem and to improve the speed and accuracy of the method, we introduce an Iterative Finite Upwind Difference Method (inspired by Luus's work in Dynamic Programming [11]) in combination with Completed Richardson Extrapolation (Richards [15]).

As is commonly known, Richardson extrapolation is a technique for improving the order of accuracy of numerical results. The main idea behind this technique is as follows. If the rate of convergence of a discretization method with grid refinement is known and if discrete solutions on two systematically refined grids (coarse and fine grids) are available, then this information can be used to provide higher-order solution on the coarse grid. As a result, it is easily implemented as a postprocessor to solutions regardless of the methods or equations producing them.

However, in order to work well, Richardson Extrapolation requires some assumptions such as smoothness and asymptotic range of the solution. Smoothness of the solution is mainly important because the analysis of Richardson Extrapolation is based on Taylor series expansion. The asymptotic range of solution means that the sequence of systematically refined grid points over which discretization error reduces at the formal order of accuracy of the discretization scheme. For example, if the order of accuracy of scheme is p, then the asymptotic range is reached as h is small (enough) such that the h^p term dominates any remaining higher-order terms. Further details can be read in Oberkampf [14].

2. Iterative Upwind Finite Difference Method

In this section we present an Iterative Upwind Finite Difference Method to solve state-constrained viscosity solution to Hamilton-Jacobi-Bellman equations. The iteration part of the method applies to the doubling of the number of discrete $x$values in order to gain better accuracy. Adjustments from iteration to iteration are designed to create efficiencies. In order to iterate without a trapezoidal propagation of the spatial domain in each time step, we impose artificial boundary conditions explained later. Firstly, we convert the state-constrained problem,

Problem 2.1.

$$min_{u(t)\in\Omega} J(u) = \int_s^{t_f} L(x(t),u(t),t)dt + \phi(\mathrm{x}(\mathrm{t_f}))$$

subject to $\dot{x} = f(x(t),u(t),t)$, *for all* $t \in (s,t_f]$,

$$x(s) = y,$$

where

$\Omega = \{u(t) \in \mathrm{R}^q \mid g(x,u,t) \le 0\}$, *for all* $t \in (0,t_f]$, $t_f > 0$ *is a constant,*

$(s,y) \in [0,t_f) \times \mathrm{R}^n$, $x \in \mathrm{R}^n$, $y \in \mathrm{R}^n$ *is a given point and* $L : R^{n+q+1} \to R, \varphi : R^n \to R, f : R^{n+q+1} \to R^n$ *and* $g : R^{n+q+1} \to R^m$ *are known functions.*

to an unconstrained optimal control problem by incorporating linear penalty terms in the objective function,

Problem 2.2.

$$\min_{u(t)\in\mathbb{R}^q} \quad P(u,r) = J(u) + \int_s^{t_f} r^T(t) g_\rho(x(t),u(t),t)dt$$

subject to $\dot{x} = f(x(t),u(t),t)$ *for all* $t \in (s,t_f]$,

$$x(s) = y,$$

where $r = (r_1(t),r_2(t),...,r_m(t))^T$ *is a vector satisfying* $r_i(t) > 0$ *for* $i=1,2,...,m$ *and* $g_\rho : R^{n+q+1} \to R$ *is the smoothed version of the constraints* $g = (g_1,g_2,...,g_m)^T$ *(see Teo* [21] *and* [23] *for more detail) defined as*

$$g_\rho(x,u,t) = \begin{cases} 0 & \text{if } g < -\rho; \\ \frac{(g+\rho)^2}{4\rho} & \text{if } -\rho \le g \le \rho; \\ g & \text{if } g > \rho. \end{cases}$$

Remark 2.1. *Note that in this case smoothing the sharp corner of the function g at zero is necessary with the aim of applying standard optimization routines.*

The corresponding HJB equation of this unconstrained problem is as follows:

$$\frac{\partial V}{\partial t} + \inf_u(\nabla V \cdot f(x,u,t) + L(x,u,t) + rg_\rho(x,u,t)) = 0 \qquad (2)$$

with terminal condition

$$V(t_f,x) = \varphi(x(t_f)).$$

Remark 2.2. *For the convergence analysis of the linear penalty on constrained viscosity solution of HJB equations, we refer to Bardi* [1]*, Bokanowski* [2] *and Loreti* [12].

In order to have a smooth solution as required by Richardson Extrapolation method, we need to change equation (2) to the singularly perturbed convectiondiffusion equation

$$\frac{\partial V}{\partial t} + \inf_u(\nabla V \cdot f(x,u,t) + L(x,u,t) + rg_\rho(x,u,t)) - \varepsilon\nabla^2 V = 0 \qquad (3)$$

with terminal condition

$$V(t_f,x) = \varphi(x(t_f)).$$

The difference between equation (2) and (3) is only the diffusion term $-\varepsilon\nabla^2 V$, $\varepsilon > 0$, which represents a small perturbation parameter. As $\varepsilon \to 0$ the solution of (3) converges to the solution (2)(see Bardi [1]).

2.1. **Discretization of HJB.** To simplify notation, let us consider an optimal control problem with one control u and one state variable $x \in [a,b]$. Extension to a multivariable optimal control problem can be easily done with some adjustments to notation. In addition, without loss of generality and for the intention of numerical tests later, we will set $s = 0$, $y = x_0$ and $t_f = 1$.

We start with constructions of spatial discretization and time stages. We select a positive integer M and divide the space interval $[a,b]$ into M equal partitions so that

$$\Delta x = \frac{b-a}{M}.$$

Therefore, the discretization for space interval becomes

$$x_i = a + (i-1)\Delta x$$

where $i = 1,\dots,M+1$.

In order that this spatial discretization always contains the initial point, an appropriate shifting might be necessary. Let $j = \arg\min_i |x_i - x_0|$ and make the adjustments

$$\begin{aligned} x_i &\leftarrow x_i + (x_0 - x(j)),\ i = 1,\dots,M \\ a &\leftarrow a + (x_0 - x(j)) \\ b &\leftarrow b + (x_0 - x(j)) \end{aligned}$$

Next, we impose a limit on control $u(t)$ where $t \in [0,1]$, specifically, the lower bound u_l and the upper bound u_u. This constraint is usually determined by physical limits of the system control values.

The time interval $[0,1]$ is then divided into N equal partitions with $\Delta t = -\frac{1}{N}$ so that

$$t_k = 1 + (k-1)\Delta t, \qquad k = 1,\dots,N+1$$

is the backward partition. This means that t_1 and t_{N+1} correspond to $t = 1$ and $t = 0$ respectively.

With notation $V_i^k \approx V(t_k, x_i)$ and $u_i^k \approx u(t_k, x_i)$ for the value function and control variable at point x_i and time t_k, we split the equation (3) into 2 equations and discretize it for $i = 1,\dots,M+1$ as follows

$$\begin{aligned} \frac{V_i^{k+1} - V_i^k}{\Delta t} &+ \frac{1+\mathrm{sign} f_i^k}{2} f_i^k \frac{V_{i+1}^k - V_i^k}{\Delta x} + \frac{1-\mathrm{sign} f_i^k}{2} f_i^k \frac{V_{i-1}^k - V_i^k}{\Delta x} + L_i^k \\ &+ r\, g_{\rho,i}^k - \varepsilon \frac{V_{i-1}^k - 2V_i^k + V_{i+1}^k}{(\Delta x)^2} = 0 \end{aligned} \tag{4}$$

$$u_i^{k+1} = \arg\inf_u (f(x_i,u,t_{k+1}) \frac{V_i^{k+1} - V_{i-1}^{k+1}}{\Delta x} + L(x_i,u,t_{k+1}) + r g_p(x_i,u,t_{k+1}$$

where $f_i^k = f(x_i,t_k,u_i^k), L_i^k = L(x_i,t_k,u_i^k), g_{\rho,i}^k = g_\rho(x_i,t_k,u_i^k)$ and sign f_i^k denotes the sign of f at point x_i and time t_k.

Using $\eta_1 = \frac{\Delta t}{\Delta x}$ and $\eta_2 = \frac{\Delta t}{(\Delta x)^2}$, equation (4) can be rewritten as follows:

$$\begin{aligned} V_i^{k+1} &= (1 + \eta_1 |f_i^k| - 2\varepsilon\eta_2) V_i^k - \left[\frac{1+\mathrm{sign} f_i^k}{2}\eta_1 f_i^k + \varepsilon\eta_2\right] V_{i+1}^k \\ &+ \left[\frac{1-\mathrm{sign} f_i^k}{2}\eta_1 f_i^k + \varepsilon\eta_2\right] V_{i-1}^k - \Delta t (L_i^k + r\, g_{\rho_i}{}^k) \end{aligned}$$

for $i = 2,\dots,M$. (5)

Because the Upwind Finite-Difference Method is an explicit method, we need to take account of the stability of the scheme under some conditions on the step length Δt and Δx.

Theorem 2.1. *Under the condition*

$$N \geq \frac{\Delta x \, \|f\|_\infty - 2\varepsilon}{(\Delta x)^2}. \tag{6}$$

scheme (5) is stable.

Proof. It is known from Strikwerda [19], that the scheme (5) is stable if and only if with $L+rg_\rho = 0$ it is also stable. The scheme (5) with $L+rg_\rho = 0$ is equivalent to

$$\begin{aligned} V_i^{k+1} &= \left[1 + \frac{1+\mathrm{sign} f_i^k}{2}\eta_1 f_i^k - \frac{1-\mathrm{sign} f_i^k}{2}\eta_1 f_i^k - 2\varepsilon\eta_2\right] V_i^k \\ &- \left[\frac{1+\mathrm{sign} f_i^k}{2}\eta_1 f_i^k + \varepsilon\eta_2\right] V_{i+1}^k + \left[\frac{1-\mathrm{sign} f_i^k}{2}\eta_1 f_i^k + \varepsilon\eta_2\right] V_{i-1}^k \end{aligned}$$

Denote the discrete maximum norm

$$\|V^k\|_\infty = \max_{1\leq i\leq M+1} |V_i^k|$$

and

$$\begin{aligned} \alpha &= -\frac{1+\mathrm{sign} f_i^k}{2}\eta_1 f_i^k + \varepsilon\eta_2 \\ \beta &= \frac{1-\mathrm{sign} f_i^k}{2}\eta_1 f_i^k + \varepsilon\eta_2 \end{aligned}$$

then, under the condition $0 < \alpha + \beta < 1$, we prove that

$$\begin{aligned} \left|V_i^{k+1}\right| &= \left|(1-\alpha-\beta)V_i^k + \alpha V_{i+1}^k + \beta V_{i-1}^k\right| \\ &\leq (1-\alpha-\beta)|V_i^k| + \alpha\,|V_{i+1}^k| + \beta|V_{i-1}^k| \\ &\leq \left\|V^k\right\|_\infty \end{aligned}$$

Taking the maximum for both sides with respect to *i* and using the induction method, we get

$$kV\ N+1k\infty \ \leq\ kV\ Nk\infty \ \leq \ldots \leq\ kV\ 1k,$$

in other words, the scheme is stable under the condition $0 < \alpha + \beta < 1$.
Furthermore, in terms of N and Δx, the above stability condition becomes

$$N \geq \frac{\Delta x \, \|f\|_\infty - 2\varepsilon}{(\Delta x)^2}.$$

Remark 2.3. *Note that the stability condition (6) is the extension of the stability condition in Wang* [24] *in the case for* ε 6= 0. Next, we set the initial value function according to

$$V_i^1 = \phi(x_i(1)),\ i = 1, \ldots, M+1$$

and initial control value for $i = 2,...,M + 1$

$$u^1_i = \text{argmin}\left(f(t_1, x_i, u)\frac{V_i^1 - V_{i-1}^1}{\Delta x} + L(t_1, x_i, u) + rg_\rho(t_1, x_i, u) \right).$$

The effect of a trapezoidal propagation on the spatial domain for each time stage can be avoided by setting up some artificial boundary conditions for control and value function based on linear extrapolation of the closest known points. The linear extrapolation to boundaries is chosen because it is simple to apply in computation and the HJB equation is a first order PDE. Moreover, it gives freedom for the edge points to flip following the line directed by the values of two closest points. Thereby, for $i = 1$

$$u_1^1 = 2u_2^1 - u_3^1.$$

Next, we update the value function for $k = 1,...,N$ and $i = 2,...,M$ according to (5) and do extrapolations for both boundaries

$$V_1^{k+1} = 2V_2^{k+1} - V_3^{k+1}.$$
$$V_{M+1}^{k+1} = 2V_M^{k+1} - V_{M-1}^{k+1}.$$

Moreover, to update control we set for $k = 1, \ldots, N$ and $i = 2, \ldots, M + 1$,

$$u_i^{k+1} = \underset{u}{\text{argmin}} \left(f(t_{k+1}, x_i, u)\frac{V_i^{k+1} - V_{i-1}^{k+1}}{\Delta x} + L(t_{k+1}, x_i, u) + r\, g_\rho(t_{k+1}, x_i, u) \right)$$

and for the left boundary

$$u_1^{k+1} = 2u_2^{k+1} - u_3^{k+1}.$$

So far we have obtained V_i^k and u_i^k for $i = 1,...,M + 1$ and $k = 1,...,N + 1$.
These constitute the first iteration of the method.

2.2. **Finding Optimal Trajectory and Control.** To iterate, we first need to determine the optimal trajectory from the first iteration. Starting with the initial value, we integrate forward the state equation $\dot{x} = f(t,x,u)$ using the following predictor-corrector method. Let us name the resultant trajectory and control y_p and u_p for predictor, y_c and u_c for corrector with $y_p(1) = y_c(1) = x_0$ and $u_c(1) = u(x_0, t_{N+1})$ respectively.

The control value used during the integration is the optimal control value corresponding to the closest grid point to the resultant state as suggested in Tremblay [22]. Thus, for $l = 2,...,N + 1$ $y_p(l) = y_c(l - 1) - \Delta t f(t_{-l+N+3}, y_p(l), u_c(l - 1))$

$$\begin{aligned} u_p(l) &= u(x_{i*}(l), t_{-l+N+3}) \\ &\quad \text{where } i^* = \arg\min_i |y_p(l) - x_i|,\ i = 1,...,M + 1 \\ y_c(l) &= y_c(l-1) - \tfrac{1}{2}\Delta t\big(f(t_{-l+N+3}, y_c(l-1), u_c(l-1)) \\ &+ f(t_{-l+N+3}, y_p(l), u_p(l))\big) \\ u_c(l) &= u(x_{i^*}, t_{-l+N+3}) \end{aligned}$$

where $i^* = \arg\min_i |y_c(l) - x_i|$, $i = 1,...,M + 1$.

The resultant pair $(y_c(l), u_c(l))$ for $l = 1,...,N + 1$ makes an optimal trajectory and control for all time steps from the first iteration of the HJB. In addition, the value function along the optimal trajectory can be determined by the value function of the corresponding closest grid points. The penalty value and objective function value can also be evaluated by forward integration along the optimal trajectory of corresponding terms.

2.3. **Region Size Reduction.** Now, we determine a procedure for region reduction based on the optimal trajectory and control from previous iteration. This new region is applied to the next iteration in order to improve computational speed and accuracy. What we need, first, is the maximum and the minimum value of resultant control and trajectory. Thus, for $l = 1, \ldots, N+1$

$$
\begin{aligned}
x\max &= \max y_c(l) \\
x\min &= \min y_c(l) \\
u\max &= \max u_c(l) \\
u\min &= \min u_c(l)
\end{aligned}
$$

In view of the fact that for the next iteration the number of interval partitions M will be doubled, we set the region for the next iteration as follows:

$$
\begin{aligned}
a &= x_{\min} - c\,\Delta x \\
b &= x_{\max} + c\,\Delta x \\
M &= 2M
\end{aligned}
$$

and the lower and upper bound for the control

$$
\begin{aligned}
u_l &= \lfloor u_{\min} \rfloor \\
u_u &= \lceil u_{\max} \rceil
\end{aligned}
$$

where u_l and u_u are consecutively the lower bound and upper bound for the control and $\lfloor z \rfloor$ means rounding the elements of z to the nearest integer less than or equal to z, $\lceil z \rceil$ rounding the elements of z to the nearest integer greater than or equal to z and c some given positive integer. c is used to make the region larger so as to improve stability.

We repeat the above steps, i.e. discretization of the computation of the HJB equation, forward integration of optimal trajectory and region reduction until iteration has nearly reached convergence. For that purpose, we may prescribe a lower bound for space interval shrinkage factor %x, i.e. the ratio of latter space interval length to former. The smaller the shrinkage factor is, the larger the reduction of the space interval length for the next iteration. The shrinkage factor close to 1 indicates that the length of space interval for the next iteration does not change much. By setting a lower bound for space interval shrinkage factor high, for instance 95%, it will ensure that the asymptotic range requirement for applying Richardson Extrapolation is satisfied. Afterwards, we run additional iteration with Completed Richardson Extrapolation on the region reduction from the last iteration. This improves the result accuracy from first-order to second-order.

3. Completed Richardson Extrapolation

The Completed Richardson Extrapolation proposed by Roache and Knupp in [16] is an extension of the original Richardson Extrapolation. They completed the method by giving higher-order solution not only on the coarse grid but on the entire fine grid. In particular, they presented application of the extrapolation on numerical solution of time-independent partial differential equations as examples. Furthermore, Richard [15] modified it in order to be used on time-dependent partial differential equation problems.

In short, the formulas for Completed Richardson Extrapolation are as follows. Let $\phi_{c,i}$ and $\phi_{f,j}$ denote respectively the first order approximate solution at node i on the coarse and j on the fine grid. The fine grid here is formed by bisecting the coarse grid such that the fine grid coincide with the coarse grid only when the indices are odd ($j = 2i-1$) where $i = 1,2,\ldots,N+1$. Then the extrapolated second order approximate solution by the Completed Richardson Extrapolation, $\phi_{RE,j}$, is determined by

$$\phi RE,j = 2\phi_{f,j} - \phi_{c,i} \text{ for } j = 2i-1 \tag{7}$$

$$\phi RE,j+1 = \phi f,j+1 + 0.5(\phi RE,j - \phi f,j + \phi RE,j+2 - \phi f,j+2) \text{ for } j+1 \text{ even} \tag{8}$$

From the last iteration, we choose M as the number of coarse grid so that

$$\begin{aligned}
M_c &= M \\
\Delta x_c &= \frac{b-a}{M_c} \\
N_c &\geq \frac{\Delta x_c \, \|f\|_\infty - 2\varepsilon}{(\Delta x_c)^2} \\
\Delta t_c &= -\frac{1}{N_c} \\
\eta_{1,c} &= \frac{\Delta t_c}{\Delta x_c} \\
\eta_{2,c} &= \frac{\Delta t_c}{(\Delta x_c)^2}
\end{aligned}$$

Analoguous with coarse grid, we can determine $\Delta x_f;\Delta t_f;\eta_{1,f};\eta_{2,f}$ using $M_f = 2M_c$ and $N_f = 2N_c$ for fine grid.

Remark 3.1. *At this stage, an appropriate shifting in the spatial discretization might be necessary to include the starting point* x_0.

Theorem 3.1. *The condition* $N_f = 2N_c$ *fulfills the stability condition in (6).*

Proof.

$$\begin{aligned}
N_f = 2N_c &= 2\left[\frac{\Delta x_c \, \|f\|_\infty - 2\varepsilon}{(\Delta x_c)^2}\right] \\
&\geq 2\left[\frac{\Delta x_c \, \|f\|_\infty - 4\varepsilon}{(\Delta x_c)^2}\right] = \left[\frac{\Delta x_f \, \|f\|_\infty - 2\varepsilon}{(\Delta x_f)^2}\right].
\end{aligned}$$

Then, we run an extra iteration as before and update the value function (V_{RE}) and control (u_{RE}) for the fine grids according to (7) and (8). The resultant pair of matrices (V_{RE},u_{RE}) is the solution of HJB equation which has a second order of accuracy.

Remark 3.2. *The use of artificial boundary conditions, i.e. linear extrapolations to the boundaries, in the algorithm does not create boundary layers. Hence, nonuniform mesh layer-adapted meshes in our case are unnecessary. For further information related to boundary layers, we refer to* [5], [13], [17], [20] *and the* *references cited therein.*

4. Numerical Experiment

To test the effectiveness of this algorithm, we take an example containing 1 state, 1 control and 1 mixed (state-control) inequality constraint from [26]. The problem is to minimize **Example 4.1.**

$$\min\left\{\int_0^1 (x^2+u^2-2u)\,dt+\tfrac{1}{2}(x(1))^2\right\}$$

subject to

$$\dot{x} = u \qquad x(0) = 0$$

$$g(x,u,t) = -(x^2+u^2-t^2-1) \le 0$$

The analytic optimal solution for this problem is $x^*(t) = t$ and $u^*(t) = 1$, so that the constraint is active for all $t \in [0,1]$. The value function for this solution is $-\frac{1}{6} \approx -0.166666667$.

The corresponding HJB initial-value problem for this example is

$$\begin{aligned} V_t + \min(uV_x + (x^2+u^2-2u) + rg_\rho(x,u,t)) &= 0 \\ V(1,x) &= \tfrac{1}{2}(x(1))^2 \end{aligned}$$

The numerical simulation is done using MATLAB R2010A and MATLAB Optimization Toolbox. We start with region $-1 \le x \le 2$ and $-2 \le u \le 2$ for the first iteration and then reduce it progressively according to our proposed method. The problem has been resolved for $\varepsilon = 10^{-10}$ and various values of M. The first four iterations are purely computed with Iterative Upwind Finite-Difference Method while the last iteration for $M = 256$ is the result of implementation of the Completed Richardson Extrapolation. The summary of our computations are given in Table 1.

it.	M	N	pen.	obj.	value	$[a,b]$	$[u_l,u_u]$	%x
1	16	11	0.2799	-0.2472	-0.1039	[-1.000, 2.000]	[-2, 2]	0.56
2	32	38	0.0429	-0.1792	-0.1466	[-0.375, 1.319]	[0, 2]	0.71
3	64	107	0.0039	-0.1646	-0.1570	[-0.106, 1.098]	[0, 2]	0.89
4	128	238	0.0107	-0.1688	-0.1600	[-0.038, 1.039]	[0, 2]	0.96
5	256	497	0.0040	-0.1659	-0.1623	[-0.017, 1.015]	[0, 2]	–

Table 1. Computational result for $e = r = 2, \varepsilon = 10^{-10}$.

The first, second and third column are respectively the number of iterations, the number of spatial and time partitions. The penalty value and objective function value along the optimal trajectory for each iteration are shown in fourth and fifth columns. These values are evaluated by forward integration of corresponding terms

along the optimal trajectory whereas the value function in sixth column is the value function at the initial point obtained from the Upwind Finite Difference Method.
The discrepancy between the objective function value and the value function in each iteration is caused by the use of state and control values of the corresponding closest grid points along the optimal trajectory in the evaluation of the objective function value. However, from iteration to iteration this discrepancy becomes smaller. This indicates that the use of the state and control values of the corresponding closest grid points is a reasonable choice. It can be seen also that in general the penalty and value function decrease as the number of iterations and M increase. Additional information related to the space and control interval used during the iteration are in seventh and eighth column. Last column contains the space interval shrinkage factor.

Tables 1, 2 and 3 indicate that the computed optimal control and state converge to the analytic solution as the error decreases significantly.

	M				
Error	16	32	64	128	256
$\lVert . \rVert_\infty$	0.1406	0.0296	0.0059	0.0068	0.0015
$\lVert . \rVert_2$	0.2346	0.0776	0.0332	0.0420	0.0194

Table 2. Computed error for u in the maximum and L_2 norm.

	M				
Error	16	32	64	128	256
$\lVert . \rVert_\infty$	0.0562	0.0083	0.0013	0.0016	0.0004
$\lVert . \rVert_2$	0.1049	0.0250	0.0069	0.0094	0.0072

Table 3. Computed error for x in the maximum and L_2 norm.

The computational results for the last iteration are plotted in the following figures. It can be seen that the value function and control shown in Figures 1 and 3 are smooth in the solution domain. This shows the success of the linear extrapolation used.

5. Concluding Remarks

In this article we present the Iterative Upwind Finite-Difference Method for the approximation of constrained viscosity solutions to Hamilton-Jacobi-Bellman Equations. As has been seen from the example, this method is very effective not only to improve the accuracy but also reduce computational time compared to the available Upwind Finite-Difference Method. In this method the trapezoidal propagation does not occur so that it reduces the number of computed grid points. Hence, the computational time is reduced.

FIGURE 1. Value function

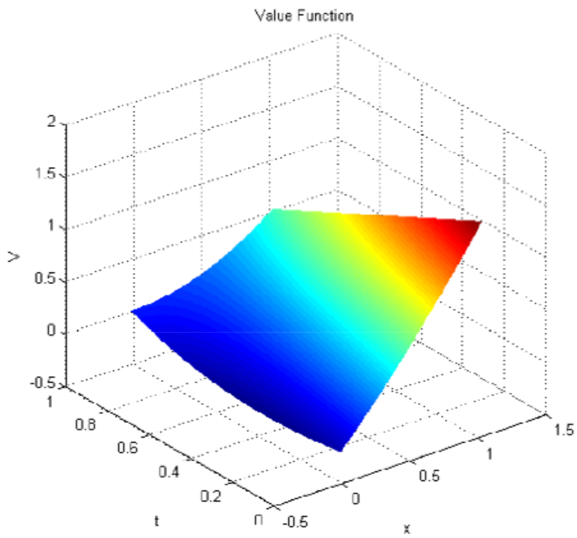
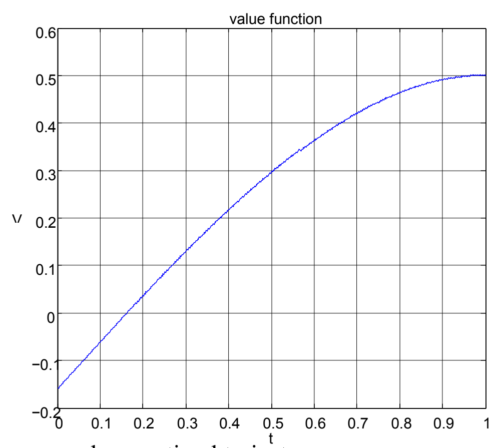


FIGURE 2. Value function



along optimal trajectory

Figure 3. Optimal control

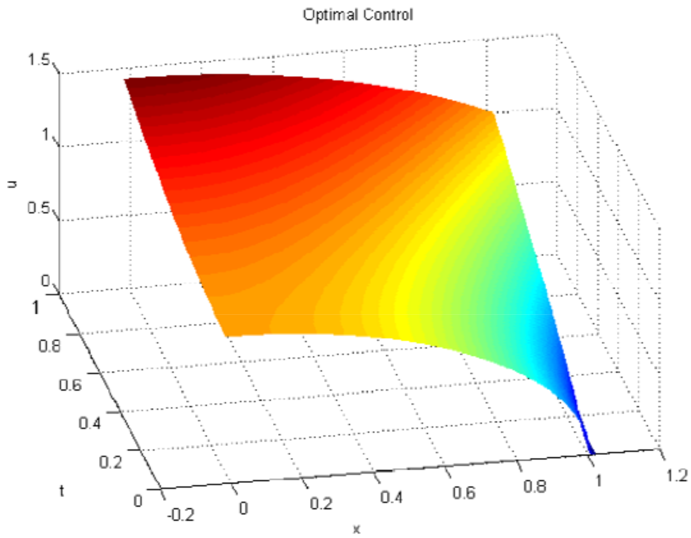


Figure 4. Optimal control along optimal trajectory

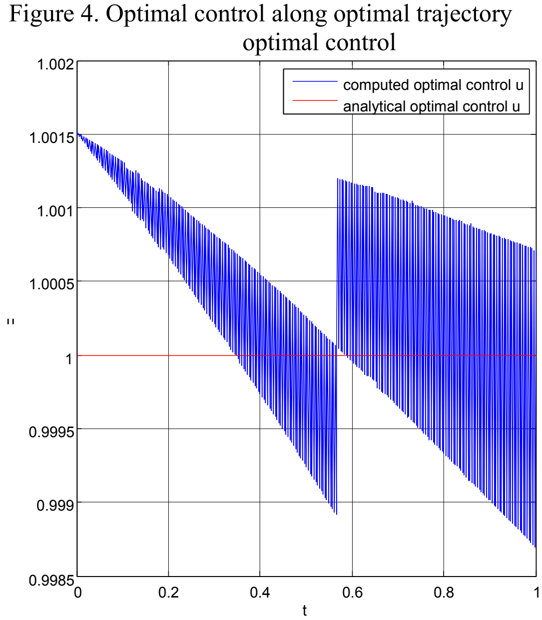


Figure 5. Optimal state along optimal trajectory

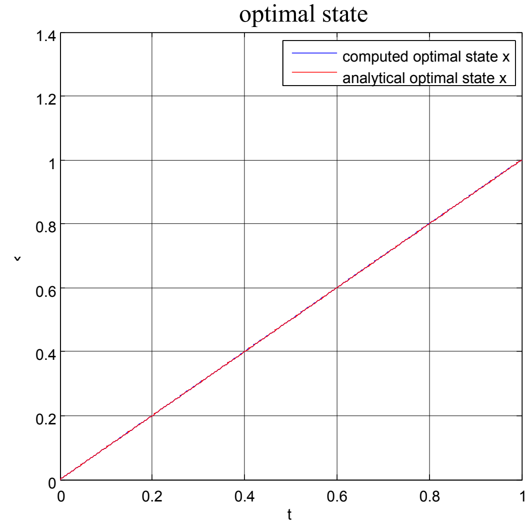


Figure 6. Constraint along optimal trajectory

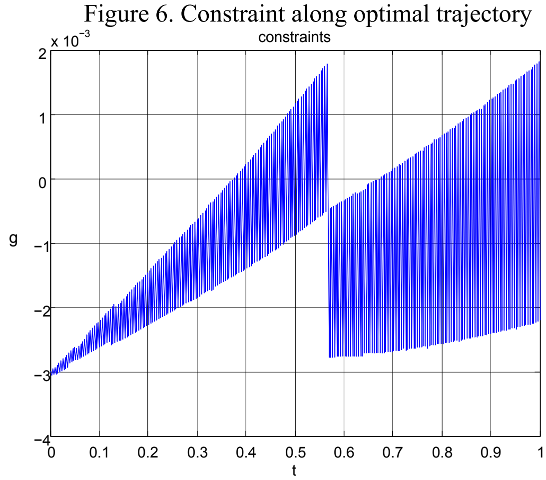
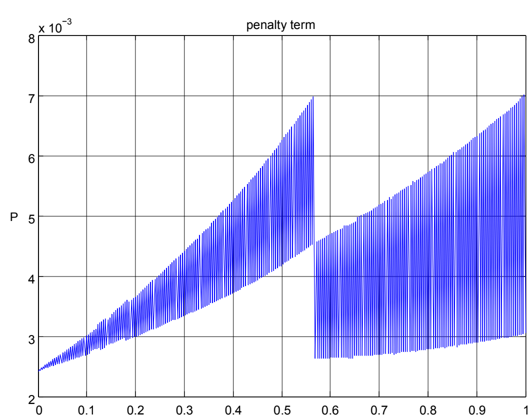


Figure 7. Penalty along optimal trajectory for Example 4.1



References

[1] M. Bardi and I. Capuzzo-Dolcetta, Optimal Control and Viscosity Solutions of HamiltonJacobi-Bellman Equations, *Birkhauser*, Boston-Basel-Berlin, (1997).

[2] O. Bokanowski, N. Forcadel and H. Zidani, Deterministic State-Constrained Optimal Control Problems without Controllability Assumptions, *ESAIM: Control, Optimisation and Calculus of Variations*, **17**, 995–1015, (2011).

[3] M. G. Crandall, L. C. Evans and P. L. Lions, Some Properties of Viscosity Solutions of Hamilton-Jacobi Equations, *Transactions of the American Mathematical Society*, **282**, 487– 502, (1984).

[4] M. G. Crandall and P. L. Lions, Viscosity Solutions of Hamilton-Jacobi Equations, *Transactions of the American Mathematical Society*, **277**, 1–42, (1983).

[5] P. A. Farrell, A. F. Hegarty, J. J. H. Miller, E. O. Riordan and G. I. Shishkin, Robust Computational Techniques for Boundary Layers, *Chapman & Hall/CRC*, (2000).

[6] W. H. Fleming and H. M. Soner, Controlled Markov Processes and Viscosity Solutions, *Springer Verlag*, New York, (1993).

[7] B. Z. Guo, B. Sun, A New Algorithm for Finding Numerical Solution of Optimal Feedback Control, *IMA Journal of Mathematical Control and Information*, **26**, 95–104, (2009).

[8] B. Z. Guo and T. T. Wu, Approximation of Optimal Feedback Control: a Dynamic Programming Approach, *Journal of Global Optimization*, **46**, 395–422, (2010).

[9] C. S. Huang, S. Wang and K. L. Teo, Solving Hamilton-Jacobi-Bellman Equations by a Modified Method of Characteristics, *Nonlinear Analysis*, **40**, 279–293, (2000).

[10] P. L. Lions, Generalised Solutions of Hamilton-Jacobi equation, *Pitman Research Notes in Mathematics*, **69**, (1982).

[11] R. Luus, Iterative Dynamic Programming, *Chapman and Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*, Boca Raton, (2000).

[12] P. Loreti and M. E. Tessitore, Approximation and Regularity Results on Constrained Viscosity Solution of Hamilton-Jacobi-Bellman Equations *Journal of Mathematical System, Estimation and Control*, **4**, 467–483 (1994).

[13] J. J. H. Miller, E. O. Riordan, G. I. Shishkin, Fitted Numerical Methods for Singular Perturbation Problems: Error Estimates in the Maximum Norm for Linear Problems in One and Two Dimensions, *World Scientific*, Singapore, (1996).

[14] W. L. Oberkampf and C. J. Roy, Verification and Validation in Scientific Computing, *Cambridge University Press*, (2010).

[15] S. A. Richards, Completed Richardson Extrapolation in Space and Time, *Communications in Numerical Methods in Engineering*, **13**, 573–582, (1997).

[16] P. J. Roache and P. M. Knupp, Completed Richardson Extrapolation, *Communications in Numerical Methods in Engineering*, **9**, 365–374, (1993).

[17] H. G. Roos, M. Stynes and L. Tobiska, Robust Numerical Methods for Singularly Perturbed Differential Equations: Convection-Diffusion-Reaction and Flow Problems, *Springer*, (2008).

[18] H. M. Soner, Optimal Control with State-Space Constraint I, *SIAM Journal on Control and Optimization*, **24**, 552–561 (1986).

[19] J. C. Strikwerda, Finite Difference Schemes and Partial Differential Equations, Second Edition, *Society of Industrial and Applied Mathematics*, Philadelphia, (2004).

[20] G. I. Shishkin and L. P. Shishkina, Difference Methods for Singular Perturbation Problems, *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*, (2009).

[21] K. L. Teo and L. S. Jennings, Nonlinear Optimal Control Problems with Continuous State Inequality Constraints, *Journal of Optimization Theory and Applications*, **63**, 1–21 (1989).

[22] M. D. Tremblay and R. Luus, Optimization of Non-Steady-State Operation of Reactors, *The Canadian Journal of Chemical Engineering*, **67**, 494–502 (1989).

[23] K. L. Teo, V. Rehbock and L. S. Jennings, A New Computational Algorithm for Functional Inequality Constrained Optimization Problems, *Automatica*, **29**, 789–792 (1993).

[24] S. Wang, F. Gao and K. L. Teo An Upwind Finite-Difference Method for the Approximation of Viscosity Solutions to Hamilton-Jacobi-Bellman Equations, *IMA Journal of Mathematical Control and Information*, **17**, 167–178 (2000).

[25] S. Wang, L. S. Jennings and K. L. Teo, Numerical Solution of Hamilton-Jacobi-Bellman Equations by an Upwind Finite Volume Method, *Journal of Global Optimization*, **27**, 177– 192, (2003).

[26] A. Q. Xing, Z. H. Chen, C. L. Wang and Y. Y. Yao, Exact Penalty Function Approach to Constrained Optimal Control Problems, *Optimal Control Applications and Methods*, **10**, 173–180 (1989).

STABILITY ANALYZE OF EQUILIBRIUM POINTS OF DELAYED SEIR MODEL WITH VITAL DYNAMICS

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Abstract An SEIR (Susceptible Exposed Invected Recovered) epidemic model with vital dynamics, incubation time length, and constant recruitment is formulated. We take incubation time length as discrete time delay parameter into bilinear incidence term of our model. With mathematical algebraic manipulation, we derive the basic reproduction number as threshold value to determine whether the disease dies out found. Then, it also determine the existence and stability of our two kind of equilibrium point, that is disease and disease free equilibrium. We also analyze the effect of our time delay as latent delay parameter to the stability of those two equilibrium point and determine whether this time delay lead to Hopf bifurcation of not.

Key words and Phrases: SEIR Model, Delay Time, Vital Dynamics, Constant Recruitment

1. Introduction

In SEIR epidemiological models with vital dynamics, class of population is divided into four subclasses that is susceptible, exposed, infectious and removed classes. where ***S*** denotes the number of individuals that are susceptible to infection,***E*** denotes the number of exposed individual, ***I*** denotes the number of individuals that are infectious, and ***R*** denotes the number of individuals that have been removed with immunity. This model use the standard bilinear incidence βSI (where $\beta > 0$ is the average number of contacts per infective per unit time) rate. For example about complete assumption of this model is given by Cappasso $[1]$.

We assume the latent period as discrete time delay parameter is constant, denoted by τ and we take it into bilinear incidence term $\beta S(t)I(t-\tau)$. The number of the susceptible individuals that become exposed at time $[t-\tau]$. Zhang, et.al has investigated the dynamics of SIR epidemic model with nonlinear incidence.

Setiawan in [2] has investigated the dynamics of SIR epidemic model with vital dynamic and also use the bilinear incidence term $\beta S(t)I(t-\tau)$. Yan and Liu, 2006 in $[3]$ investigated a class of SEIR epidemiological models under assumption that there is a probability value that individuals survives the latent period $[t-\tau]$ is $e^{-\mu\tau}$, where μ is per capita death rate due to causes other than the disease. The different of model in $[3]$ with model in this paper is assumption that all of the individuals will be survives in the latent period and becoming infective at time *t*.

We arrange our paper as follows. In Section 2, we give the model formulation of our problem and its initial condition. By mathematical analysis, we prove that the existence of our equilibrium are completely determined by the values of the threshold value R_0. The stability of disease free equilibrium is also determined by R_0 and we can conclude that $R_0 = 1$ implies that E_0 is always locallyasymptotically stable stable, and there is no effect of presence of time delay to the stability of disease free equilibrium, in other words there is no bifurcation of E_0. Finally we also study the stability of endemic equilibrium.

2. Main Results

2.1. Mathematical Formulation

We consider the common SEIR model with bilinear incidence or we can say SEIR model with vital dynamics. The detail of those model can be found in many references by many authors, as example by Capasso, in $[1]$. We assume the latent period as discrete time delay parameter is constant, denoted by τ and we take it into bilinear incidence term $\beta S(t)I(t-\tau)$.The number of the susceptible individuals that become exposed at time $t-\tau$. We assume that all of the individual will be survive in the latent period and becoming infective at time *t*. Finally, we can get SEIR epidemiological model with vital dynamic and discrete time delay as follows :

$$
\begin{aligned}
\dot{S}(t) &= \delta - \delta.S(t) - \beta S(t)I(t-\tau) \\
\dot{E}(t) &= \beta S(t)I(t-\tau) - \varepsilon E(t) - \delta E(t) \qquad (2.1) \\
\dot{I}(t) &= \varepsilon E(t) - \gamma I(t) - \delta.I(t) \\
\dot{R}(t) &= \gamma.I(t) - \delta.R(t)
\end{aligned}
$$

$N(t) = S(t) + E(t) + I(t) + R(t),$

where $\delta, \beta, \tau, \varepsilon, \gamma$ are real constants parameter, where δ is the parameter that measures the natural mortality rate, β is average number of contacts per infective per unit time, τ is latent delay time, $\frac{1}{\varepsilon}$ is average latent period and γ is the natural recovery rate that of the infective individuals.

The initial conditions $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ of (2.1) are defined in the Banach space,

$$C_+ = \{\varphi \in C([-\tau,0], \mathbb{R}_+^4) : \varphi_1(\theta) = S(\theta), \varphi_2(\theta) = E(\theta), \varphi_3 = I(\theta), \varphi_4 = R(\theta), \theta \in [-\tau,0]\} \quad (2.2)$$

where $\mathbb{R}_+^3 = \{(S,E,I,R) \in \mathbb{R}^4 : S \geq 0, E \geq 0, I \geq 0, R \geq 0\}$ and by a biological meaning, we assume that $\varphi_i \geq 0$, i = 1,2,3. Because total population size *N*(t) is constant, then for convenience, we may assume that *N*(t)= *S*(t)+*E*(t)+*I*(t)= 1.

2.2 Equilibrium States and The Basic Reproduction Number

In this section we study the existence of the equilibrium points of system (2.1) by the following theorem

Theorem 2.1. *System (2.1) always disease free equilibrium* $\hat{E}_0 = (1,0,0,0)$. *Let* $R_0 = \frac{\beta\varepsilon}{(\varepsilon+\delta)(\gamma+\delta)}$. *We can get the other equilibrium based on the value of* R_0.

(1) If $R_0 = 1$ *there is one additional equilibrium point, that is the other disease free equilibrium* $E_0 = \left(\frac{(\varepsilon+\delta)(\gamma+\delta)}{\beta\varepsilon}, 0,0,0\right)$

(2) If $R_0 > 1$ *there is one additional equilibrium point, that is disease equilibrium,* $E_1 = (S^*, E^*, I^*, R^*)$, *where*

$$S^* = \frac{(\varepsilon+\delta)(\gamma+\delta)}{\beta\varepsilon}, \quad E^* \frac{(\gamma+\delta)\delta}{(\varepsilon+\delta)(\gamma+\delta)}\left(1-\frac{1}{R_0}\right), \quad I^* = \frac{\delta\varepsilon}{(\varepsilon+\delta)(\gamma+\delta)}\left(1-\frac{1}{R_0}\right),$$

$$R^* = \frac{\gamma\varepsilon}{(\varepsilon+\delta)(\gamma+\delta)}\left(1-\frac{1}{R_0}\right)$$

PROOF.

Equilibrium points is the states of a system that are satisfied the following conditions $\dot{S}(t) = 0$, $\dot{I}(t) = 0$ and $\dot{R}(t) = 0$. From system (2.1) we get

$$\begin{aligned}
&\delta - \delta.S(t) - \beta S(t)I(t-\tau) = 0 \\
&\beta S(t)I(t-\tau) - \varepsilon E(t) - \delta E(t) = 0 \\
&\varepsilon E(t) - \gamma I(t) - \delta.I(t) = 0 \\
&\gamma.I(t) - \delta.R(t) = 0
\end{aligned} \quad (2.3)$$

It's easy to prove that from equation (2.3), we can get the first disease free equilibrium $\hat{E}_0 = (1,0,0,0)$. Based on third and fourth equation of (2.3) we get

$$R^* = \frac{\gamma}{\delta} I^* \quad \text{and} \quad E^* = \frac{(\gamma+\delta)}{\varepsilon} I^*$$

Then, if we substitute E^* to first equation of (2.3) we also get

$$S^* = \frac{(\varepsilon+\delta)(\gamma+\delta)}{\varepsilon\beta} I^*$$

Finally, if we substitute S^*, E^* and R^* back to first equation of (2.3) and then by algebraic manipulation we get

$$I^* = \frac{\delta\varepsilon}{(\varepsilon+\delta)(\gamma+\delta)}\left(1-\frac{1}{R_o}\right)$$

where $R_0 = \dfrac{\beta\varepsilon}{(\varepsilon+\delta)(\gamma+\delta)}$. If $R_0 > 1$ then we have an endemic equilibrium $E_1 = \left(S^*, E^*, I^*, R^*\right)$ with

$$S^* = \frac{(\varepsilon+\delta)(\gamma+\delta)}{\varepsilon\beta} \quad E^* = \frac{(\gamma+\delta)\delta}{(\varepsilon+\delta)(\gamma+\delta)}\left(1-\frac{1}{R_0}\right)$$

$$I^* = \frac{\delta\varepsilon}{(\varepsilon+\delta)(\gamma+\delta)}\left(1-\frac{1}{R_0}\right) \quad R^* = \frac{\gamma\varepsilon}{(\varepsilon+\delta)(\gamma+\delta)}\left(1-\frac{1}{R_0}\right)$$

On the otherhand, if $R_0 = 1$ then we have an additional disease free equilibrium $E_0 = \left(\frac{(\varepsilon+\delta)(\gamma+\delta)}{\varepsilon\beta}, 0, 0, 0\right)$

Q.E.D.

2.3 Linearized System

Let $\hat{E} = \left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)$ be any equilibrium of system (2.1). Then, we will find linearized system of system (2.1) at $\hat{E} = \left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)$ by linearizing about $\hat{E} = \left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)$ with Taylor Series Method. We can write system (2.1) in matrix form as follows :

$$\begin{pmatrix} \dot{S}(t) \\ \dot{E}(t) \\ \dot{I}(t) \\ \dot{R}(t) \end{pmatrix} = \begin{pmatrix} \delta - \delta S(t) \\ -(\varepsilon+\delta)E(t) \\ \varepsilon E(t) - (\gamma+\delta)I(t) \\ \gamma I(t) - \delta R(t) \end{pmatrix} + \begin{pmatrix} -\beta S(t) I(t-\tau) \\ \beta S(t) I(t-\tau) \\ 0 \\ 0 \end{pmatrix} \quad (2.4)$$

let

$$\begin{pmatrix} f_1\left(S(t), I(t), E(t), R(t)\right) \\ f_2\left(S(t), I(t), E(t), R(t)\right) \\ f_3\left(S(t), I(t), E(t), R(t)\right) \\ f_4\left(S(t), I(t), E(t), R(t)\right) \end{pmatrix} = \begin{pmatrix} \delta - \delta S(t) \\ -(\varepsilon+\delta)E(t) \\ \varepsilon E(t) - (\gamma+\delta)I(t) \\ \gamma I(t) - \delta R(t) \end{pmatrix},$$

$$\begin{pmatrix} g_1\left(S(t), I(t-\tau), E(t), R(t)\right) \\ g_2\left(S(t), I(t-\tau), E(t), R(t)\right) \\ g_3\left(S(t), I(t-\tau), E(t), R(t)\right) \\ g_4\left(S(t), I(t-\tau), E(t), R(t)\right) \end{pmatrix} = \begin{pmatrix} -\beta S(t) I(t-\tau) \\ \beta S(t) I(t-\tau) \\ 0 \\ 0 \end{pmatrix}, \qquad (2.5)$$

$x(t) = S(t) - \hat{S}$, $y(t) = E(t) - \hat{E}$, $z(t) = I(t) - \hat{I}$, $z(t-\tau) = I(t-\tau) - \hat{I}$
$v(t) = R(t) - \hat{R}$

Then, we will determine Taylor series of
$f_1(S(t), E(t), I(t), R(t))$, $f_2\left(S(t), E(t), I(t), R(t)\right)$ and $f_3\left(S(t), E(t), I(t), R(t)\right)$
at $\hat{R} = \left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)$as follows

$$\begin{aligned} f_1\left(S(t), E(t), I(t), R(t)\right) &= f_1\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right) + \frac{\partial f_1}{\partial S(t)}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(S(t) - \hat{S}\right) + \frac{\partial f_1}{\partial E(t)}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(E(t) - \hat{E}\right) + \\ &+ \frac{\partial f_1}{\partial I(t)}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(I(t) - \hat{I}\right) + \frac{\partial f_1}{\partial R(t)}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(R(t) - \hat{R}\right) + \\ &+ \frac{1}{2}\frac{\partial^2 f_1}{\partial S(t)^2}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(S(t) - \hat{S}\right)^2 + \frac{1}{2}\frac{\partial^2 f_1}{\partial E(t)^2}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(E(t) - \hat{E}\right)^2 + \frac{1}{2}\frac{\partial^2 f_1}{\partial I(t)^2}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(I(t) - \hat{I}\right)^2 + \\ &+ \frac{1}{2}\frac{\partial^2 f_1}{\partial R(t)^2}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(R(t) - \hat{R}\right)^2 + \cdots \\ &= f_1\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right) - \delta x(t) + O\left(\left\|\left(\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\right) - \left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\right\|^2\right), \end{aligned}$$

$$\begin{aligned} f_2\left(S(t), E(t), I(t), R(t)\right) &= f_2\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right) + \frac{\partial f_2}{\partial S(t)}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(S(t) - \hat{S}\right) + \frac{\partial f_2}{\partial E(t)}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(E(t) - \hat{E}\right) + \\ &+ \frac{\partial f_2}{\partial I(t)}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(I(t) - \hat{I}\right) + \frac{\partial f_2}{\partial R(t)}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(R(t) - \hat{R}\right) + \\ &+ \frac{1}{2}\frac{\partial^2 f_2}{\partial S(t)^2}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(S(t) - \hat{S}\right)^2 + \frac{1}{2}\frac{\partial^2 f_2}{\partial E(t)^2}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(E(t) - \hat{E}\right)^2 + \frac{1}{2}\frac{\partial^2 f_2}{\partial I(t)^2}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(I(t) - \hat{I}\right)^2 + \\ &+ \frac{1}{2}\frac{\partial^2 f_2}{\partial R(t)^2}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(R(t) - \hat{R}\right)^2 + \cdots \\ &= f_2\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right) - (\varepsilon + \delta) y(t) + O\left(\left\|\left(\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\right) - \left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\right\|^2\right), \end{aligned}$$

$$\begin{aligned} f_3\left(S(t), E(t), I(t), R(t)\right) &= f_3\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right) + \frac{\partial f_3}{\partial S(t)}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(S(t) - \hat{S}\right) + \frac{\partial f_3}{\partial E(t)}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(E(t) - \hat{E}\right) + \\ &+ \frac{\partial f_3}{\partial I(t)}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(I(t) - \hat{I}\right) + \frac{\partial f_3}{\partial R(t)}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(R(t) - \hat{R}\right) + \\ &+ \frac{1}{2}\frac{\partial^2 f_3}{\partial S(t)^2}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(S(t) - \hat{S}\right)^2 + \frac{1}{2}\frac{\partial^2 f_3}{\partial E(t)^2}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(E(t) - \hat{E}\right)^2 + \frac{1}{2}\frac{\partial^2 f_3}{\partial I(t)^2}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(I(t) - \hat{I}\right)^2 + \\ &+ \frac{1}{2}\frac{\partial^2 f_3}{\partial R(t)^2}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(R(t) - \hat{R}\right)^2 + \cdots \\ &= f_3\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right) + \varepsilon y(t) - (\gamma + \delta) z(t) + O\left(\left\|\left(\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\right) - \left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\right\|^2\right), \end{aligned}$$

$$\begin{aligned} f_4\left(S(t), E(t), I(t), R(t)\right) &= f_4\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right) + \frac{\partial f_4}{\partial S(t)}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(S(t) - \hat{S}\right) + \frac{\partial f_4}{\partial E(t)}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(E(t) - \hat{E}\right) + \\ &+ \frac{\partial f_4}{\partial I(t)}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(I(t) - \hat{I}\right) + \frac{\partial f_4}{\partial R(t)}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(R(t) - \hat{R}\right) + \\ &+ \frac{1}{2}\frac{\partial^2 f_4}{\partial S(t)^2}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(S(t) - \hat{S}\right)^2 + \frac{1}{2}\frac{\partial^2 f_4}{\partial E(t)^2}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(E(t) - \hat{E}\right)^2 + \frac{1}{2}\frac{\partial^2 f_4}{\partial I(t)^2}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(I(t) - \hat{I}\right)^2 + \\ &+ \frac{1}{2}\frac{\partial^2 f_4}{\partial R(t)^2}\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\left(R(t) - \hat{R}\right)^2 + \cdots \\ &= f_4\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right) + \gamma z(t) - \delta v(t) + O\left(\left\|\left(\left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\right) - \left(\hat{S}, \hat{E}, \hat{I}, \hat{R}\right)\right\|^2\right) \end{aligned}$$

Then, by same method we can get

$$g_1\big(S(t),E(t),I(t-\tau),R(t)\big) = g_1\big(\hat{S},\hat{E},\hat{I},\hat{R}\big) + \frac{\partial g_1}{\partial S(t)}\big(\hat{S},\hat{E},\hat{I},\hat{R}\big)\big(S(t)-\hat{S}\big) + \frac{\partial g_1}{\partial E(t)}\big(\hat{S},\hat{E},\hat{I},\hat{R}\big)\big(E(t)-\hat{E}\big) +$$

$$+ \frac{\partial g_1}{\partial I(t)}\big(\hat{S},\hat{E},\hat{I},\hat{R}\big)\big(I(t-\tau)-\hat{I}\big) + \frac{\partial g_1}{\partial R(t)}\big(\hat{S},\hat{E},\hat{I},\hat{R}\big)\big(R(t)-\hat{R}\big) +$$

$$+ \frac{1}{2}\frac{\partial^2 g_1}{\partial S(t)^2}\big(\hat{S},\hat{E},\hat{I},\hat{R}\big)\big(S(t)-\hat{S}\big)^2 + \frac{1}{2}\frac{\partial^2 g_1}{\partial E(t)^2}\big(\hat{S},\hat{E},\hat{I},\hat{R}\big)\big(E(t)-\hat{E}\big)^2 + \frac{1}{2}\frac{\partial^2 g_1}{\partial I(t)^2}\big(\hat{S},\hat{E},\hat{I},\hat{R}\big)\big(I(t-\tau)-\hat{I}\big)^2$$

$$+ \frac{1}{2}\frac{\partial^2 g_1}{\partial R(t)^2}\big(\hat{S},\hat{E},\hat{I},\hat{R}\big)\big(R(t)-\hat{R}\big)^2 + \cdots$$

$$= g_1\big(\hat{S},\hat{E},\hat{I},\hat{R}\big) - \beta\hat{I}x(t) - \beta\hat{S}z(t-\tau) + O\left(\left\|\big(S,E,I,R\big) - \big(\hat{S},\hat{E},\hat{I},\hat{R}\big)\right\|^2\right)$$

analogue,

$$g_2\big(S(t),E(t),I(t-\tau),R(t)\big) = g_2\big(\hat{S},\hat{E},\hat{I},\hat{R}\big) + \beta\hat{I}x(t) + \beta\hat{S}z(t-\tau) + O\left(\left\|\big(\hat{S},\hat{E},\hat{I},\hat{R}\big) - \big(\hat{S},\hat{E},\hat{I},\hat{R}\big)\right\|^2\right)$$

and $g_3\big(S(t),I(t-\tau),E(t),R(t)\big) = g_4\big(S(t),I(t-\tau),E(t),R(t)\big) = 0$. So, we get

linearized system of system (2.1)as follows :

$$\dot{x}(t) = \big(-\delta - \beta\hat{I}\big)x(t) - \beta\hat{S}z(t-\tau)$$

$$\dot{y}(t) = -(\varepsilon + \delta)y(t) + \beta\hat{I}x(t) + \beta\hat{S}z(t-\tau)$$

$$\dot{z}(t) = \varepsilon y(t) \quad (\gamma + \delta)z(t)$$

$$\dot{v}(t) = \gamma z(t) - \delta v(t)$$

$$(2.6)$$

Finally, we can find the roots of system (2.6) by solving the following equation

$$det\begin{pmatrix} -\big(\delta+\beta\hat{I}\big)-\lambda & 0 & -\beta\hat{S}e^{-\lambda\tau} & 0 \\ \beta\hat{I} & -(\varepsilon+\delta)-\lambda & \beta\hat{S}e^{-\lambda\tau} & 0 \\ 0 & \varepsilon & -(\gamma+\delta)-\lambda & 0 \\ 0 & 0 & \gamma & -\delta-\lambda \end{pmatrix} = 0$$

$$(2.7)$$

2.4 Stability of Disease Free Equilibrium

In this section, we study the local stability of disease free equilibrium of system (2.1) and the effect of time delay to the stability of disease free equilibrium.If we evaluate equation (2.7) at E_0 then we have the following equation

$$\det\begin{pmatrix} -\delta-\lambda & 0 & -\beta\dfrac{(\varepsilon+\delta)(\gamma+\delta)}{\varepsilon\beta}e^{-\lambda\tau} & 0 \\ 0 & -(\varepsilon+\delta)-\lambda & \beta\dfrac{(\varepsilon+\delta)(\gamma+\delta)}{\varepsilon\beta}e^{-\lambda\tau} & 0 \\ 0 & \varepsilon & -(\gamma+\delta)-\lambda & 0 \\ 0 & 0 & \gamma & -\delta-\lambda \end{pmatrix} = 0 \tag{2.8}$$

one of root of equation (2.8) is always $\lambda = -\delta$ which is independent of any parameters. We can get the others roots of equation (2.8) from the following equation

$$(\varepsilon+\delta+\lambda)(\gamma+\delta+\lambda) - (\varepsilon+\delta)(\gamma+\delta)e^{-\lambda\tau} = 0$$

For $\tau = 0$, we get

$$\lambda^2 + (\varepsilon+\gamma+2\delta)\lambda = 0$$

Based from the above equation, we can conclude that $\tau = 0$ then all of the value of roots of equation (2.8) are negative, therefore E_0 is asymptotically stable. In this paper we do not analyze E_0 for $\tau \neq 0$ and we left this problem to further research.

2.5 Stability of DiseaseEquilibrium

In this section, we study the local stability of disease equilibrium E_1. If we evaluate equation (2.7) at E_0 then we have the following equation

$$\left(\delta + \beta\frac{\delta\varepsilon}{(\varepsilon+\delta)(\gamma+\delta)}\left(1-\frac{1}{R_0}\right)+\lambda\right)(\varepsilon+\delta+\lambda)(\gamma+\delta+\lambda) - (\delta+\lambda)^2(\varepsilon+\delta)(\gamma+\delta)e^{-\lambda\tau} = 0 \tag{2.9}$$

Let $A = \varepsilon + \delta$ and $B = \gamma + \delta$, then we rewrite equation (2.9) into

$$(\lambda+\delta)\left(\left(\delta + \frac{\beta\delta\varepsilon}{AB}\left(1-\frac{1}{R_o}\right)+\lambda\right)(\lambda+A)(\lambda+B) - (\lambda+\delta)ABe^{-\lambda\tau}\right) = 0 \tag{2.10}$$

or equivalent with

$$\frac{\lambda^3}{AB} + \left[2+\frac{\delta C}{AB}\left(1-\frac{1}{R_0}\right)\right]\lambda^2 + \left[1+2\delta C\left(1-\frac{1}{R_0}\right)\right]\lambda + \delta C\left(1-\frac{1}{R_0}\right) - (\lambda+\delta)e^{-\lambda\tau} = 0 \tag{2.11}$$

where $C = 1 + \dfrac{\beta\varepsilon}{AB}$

Theorem 2.2.

Let $R_c = \dfrac{2\delta C}{(2\delta C - 1)}$, *which is* $R_c > 1$. *If* $R_0 > R_c$, *then* E_1 *is locallyasymptotically stable stable for* $\tau = 0$

PROOF.

Based on equation (2.11) for $\tau = 0$, we will get the following equation

$$\frac{\lambda^3}{AB} + \left[2 + \frac{\delta C}{AB}\left(1 - \frac{1}{R_0}\right)\right]\lambda^2 + \left[2\delta C\left(1 - \frac{1}{R_0}\right)\right]\lambda + \delta C\left(1 - \frac{1}{R_0}\right) - \delta = 0 \quad (2.12)$$

If we use Routh-Hurwitz stability criterion, especially for a fourth-order polynomial, to determine the sign of all the roots of equation (2.12). Let

$$a_3 = \frac{1}{AB} \; a_2 = \left[2 + \frac{\delta C}{AB}\left(1 - \frac{1}{R_0}\right)\right]$$
$$a_1 = \left[2\delta C\left(1 - \frac{1}{R_0}\right)\right] a_0 = \delta C\left(1 - \frac{1}{R_0}\right) - \delta$$

It's easy to prove that equation (2.12) is satisfy $a_n > 0, n = 0,1,2,3,4$ and also $a_2 a_1 > a_3 a_0$, so all the roots of equation (2.12) have negative sign. **Q.E.D.**

Then, we will study the effect of time delay to the stability of disease equilibrium (case $\tau \neq 0$). Without loss of generality, if $\lambda = i\omega$, $\omega \in R$ then, if we substitute to the equation (2.9) we get

$$\begin{aligned} i\omega^3 - (A+B)\omega^2 + i\omega AB + \delta C\left(1 - \frac{1}{R_0}\right)\omega + i\delta C\left(1 - \frac{1}{R_0}\right)(A+B)\omega + \delta ABC\left(1 - \frac{1}{R_0}\right) \\ - i\omega AB\cos(\omega\tau) - \delta AB\cos(\omega\tau) + \omega AB\sin(\omega\tau) - \delta AB\, i\sin(\omega\tau) = 0 \end{aligned}$$

(2.13)

By separating the real dan imaginary parts, and then squaring and adding those both equations, then we have the following equation :

$$\begin{aligned} &((A+B)^2+1)\omega^4 + 2AB\omega^3 \\ &\quad + \left[\left(1 - \frac{1}{R_0}\right)^2((A+B)^2+1)(\delta C)^2 - 2(A+B)\delta ABC\left(1 - \frac{1}{R_0}\right)\right]\omega^2 \\ &\quad + 2\delta^2C^2AB\left(1 - \frac{1}{R_0}\right)^2\omega + (\delta ABC)^2\left(1 - \frac{1}{R_0}\right)^2 - (\delta AB)^2 = 0 \end{aligned}$$

(2.14)

It's difficult to get direct conditions to get real solutions of equation (2.14) and we left this problem to further research.

3. Concluding Remarks

From system (2.1) we get two equilibrium, that is disease free equilibrium and endemic equilibrium. If $R_0 = 1$ implies that E_0 is always locallyasymptotically stable stable, and there is no effect of presence of time delay to the stability of disease free equilibrium, in other words there is no bifurcation of E_0. Then, if

$R_c = \frac{2\delta C}{\left(2\delta C - 1\right)}$, which is $R_c > 1$, E_1 is locallyasymptotically stable stable for $\tau = 0$ if and only if $R_0 > R_c$.

References

[1] Cappaso, V,1993,**Mathematical Structures of Epidemic Systems** ,Vol 97 Lecture-Notes in Biomathematics, Springer – Verlag Berlin – Heidelberg.

[2] Setiawan, R.,2009, **Stability of Delayed S I R Model With Vital Dynamics**, *Proc. IndoMS International Conference on Mathematics and Its Application* (IICMA)(2009), 471-478.

[3] Yan, P. and Liu, S.,2006, **SEIR Epidemic Model with Delay**, *ANZIAM Journal* 48 (2006), pp. 119-134.

[4] Zhang, J Z, Jin, Z, Liu,QX, Zhang, ZY, **Analysis of a Delayed SIR Model with Nonlinear Incidence,** Discrete Dynamics in Nature and Society, Vol.2008, Article ID 636153(2008), Hindawi Publishing Company.

THE TOTAL VERTEX IRREGULARITY STRENGTH OF A CANONICAL DECOMPOSABLE GRAPH, $G = S(A,B) \circ tK_1$

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Abstract. A vertex-irregular total k-labeling $\lambda : V(G) \cup E(G) \to \{1,2,\cdots,k\}$ of a graph G is a labeling of the vertices and the edges of G in such a way that for any different vertices v and w, their weights $wt(v)$ and $wt(w)$ are distinct. The weight of a vertex v is the sum of the label of v and the labels of all edges incident with v. The total vertex irregularity strength of G, denoted by $tvs(G)$, is defined as the minimum k for which a graph G has a vertex-irregular total k-labeling. In this paper, we consider a canonical decomposable graph. A canonical decomposable graph $G = S(A,B) \circ tK_1$ is a graph formed by taking the disjoint union of two graphs, the first is a split graph S whose an independent set A and a clique B, and the second is t copies of K_1, and adding to it all edges having an endpoint in B and the other endpoint in $V(tK_1)$. In this paper we determine the total vertex irregularity strength of G.

Keywords and Phrases: canonical decomposable graph, total vertex irregularity strength, vertex-irregular total k-labeling

1. Introduction

All graphs in this paper are simple, finite and undirected. Bača et al. [2] introduced a vertex-irregular total k-labeling $\lambda : V(G) \cup E(G) \to \{1,2,\cdots,k\}$ of a graph G as a labeling of the vertices and edges of G such that for every pair distinct vertices v and w, $wt(v) = \lambda(v) + \sum_{vx \in E(G)} \lambda(vx) \neq \lambda(w) + \sum_{wy \in E(G)} \lambda(wy) = wt(w)$. The *total vertex irregularity strength of G*, denoted by $tvs(G)$, is the minimum positive integer k for which G has a vertex-irregular total k-labeling.

Bača et al. [2] have determined the total vertex irregularity strength for some classes of graphs, namely cycles, stars, and prisms. Whereas the total vertex irregularity strength of trees and disjoint union of t copies of path had been determined by Nurdin et al. in [3] and [5]. Furthermore, Nurdin et al. [4] determined the total vertex irregularity strength for several types of trees containing vertices of degree 2, namely a subdivision of a star and a subdivision of a particular caterpillar. They also derived the total vertex irregularity strength for a complete $k - ary$ tree. Wijaya et al. [8] determined the total vertex irregularity strength of complete bipartite graphs. In [6], Tong et al. determined the total vertex irregularity strength of toroidal grid $C_m \square C_n$. Besides that, Al-Mushayt et al. [1] determined the total vertex irregularity strength for convex polytope graphs. In this paper, we determine the total vertex irregularity strength of a canonical decomposable graph, $G = S(A,B) \circ tK_1$.

2. Main Results

A *clique* in a graph is a set of pairwise adjacent vertices; an *independent set* is a set of pairwise non-adjacent vertices. A *split graph* $S(A,B)$ is a graph whose vertex set can be partioned into an independent set A and a clique B.

Let a,b in $V(G)$. The graph $G + ab$ is a graph obtained from G by adding a new edge ab. Let Σ and Γ denote the sets of split graphs and of simple graphs, respectively. Define the composition $\circ : \Sigma \times \Gamma \to \Gamma$ as follows:

$$\text{if } \sigma \in \Sigma,\ \sigma = S(A,B),\ H \in \Gamma \text{ then}$$
$$\sigma \circ H = (S(A,B) \cup H) + \{bc : b \in B, c \in V(H)\}.$$

(The edge set of the complete bipartite graph with parts B and $V(H)$ is added to the disjoint union $S(A,B) \cup H$). A graph G is called decomposable, if $G = \sigma \circ H$ for some $\sigma \in \Sigma$ and for some $H \in \Gamma$. Otherwise, G is indecomposable [7].

In this paper, we consider t copies of K_1 as H.

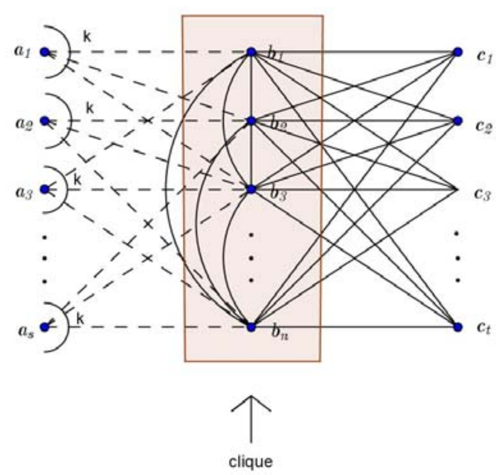

Figure 1. $G = S(A,B) \circ tK_1$ where $deg(a) = k$ with $1 \le k \le n - 1$

We use the concept of multisets. Define a multiset $X = \{a_1^{k_1}, a_2^{k_2}, \ldots, a_n^{k_n}\}$ as a number family containing k_i numbers of a_i for $i \in \{1,2,...,n\}$, $n \in \mathbf{N}$. Let X,Y be two multisets, then the multiset $X \cup Y = \{\alpha : \alpha \in X; \alpha \in Y\}$. Notice that if α appears r times in X and s times in Y, $r,s \in \{0,1,2,...\}$, then α appears $r+s$ times in $X \cup Y$.

Let X be a multiset containing positive integer. X is said to be anti balanced if there exist submultisets $X_1, X_2, \ldots, X_k$ for some $k \in \mathbb{N}$ such that $|X_i| = |X_j|$,

$\sum X_i \neq \sum X_j$, $i \neq j$, $i,j \in \{1,2,\ldots,k\}$, and $\biguplus_{i=1}^{K} X_i = X$.

Lemma 2.1. *Let n,s, and t be integers at least* 2*, z be an integer at least* 1*, and* $k \in \{1,2,\ldots,n-1\}$. *Let m and* $z+o$ *be two integers at most*

$$\begin{cases} \lceil \frac{k+s}{k+1} \rceil, & n \geq \frac{t(k+1)}{s}; \\ \lceil \frac{n+t}{n+1} \rceil, & n \geq k+s \text{ and } n < \frac{t(k+1)}{s}; \\ \lceil \frac{k+s+t}{n+1} \rceil, & n < k+s \text{ and } n < \frac{t(k+1)}{s}, \end{cases}$$

and two multisets

$$X = \biguplus_{i=1}^{k+1} \{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\}$$

$$Y = \biguplus_{j=k+2}^{k+n+2} \{z^{k_1}, (z+1)^{k_2}, \ldots, (z+o)^{k_{o+1}}\}.$$

Then there exist submultisets $X_1, X_2, \ldots, X_s$ *and* $Y_{s+1}, Y_{s+2}, \ldots, Y_{s+t}$ *such that*

- *for each* $i \in \{1,2,\ldots,s\}$, $|X_i| = k+1$;
- *for each* $j \in \{s+1, s+2, \ldots, s+t\}$, $|Y_j| = n+1$;
- $\sum X_k \neq \sum X_l$, $k \neq l$, $k,l \in \{1,2,\ldots,s\}$, $\sum Y_p \neq \sum Y_q$, $p \neq q$, $p,q \in \{s+1, s+2, \ldots, s+t\}$; *and*
- $\biguplus_{i=1}^{s} X_i = X$, $\biguplus_{i=s+1}^{s+t} Y_i = Y$.

So X and Y is anti-balanced.

Proof. For each $i \in \{1,2,\ldots,s\}$ define multiset $X_i = \{a_i b_{(1)(i)}, a_i b_{(2)(i)}, \ldots, a_i b_{(k)(i)}, a_i\}$ where for $1 \leq l \leq k$:

$$a_i b_{(l)(i)} = \begin{cases} 1, & \text{if } 1 \leq i \leq l; \\ m, & \text{if } (m-2)k + m + l - 1 \leq i \leq (m-1)k + m + l - 1 \end{cases}$$

and

$$a_i = \begin{cases} 1, & \text{if } 1 \leq i \leq k+1; \\ m, & \text{if } (m-1)k + m \leq i \leq mk + m. \end{cases}$$

For each $j \in \{s+1, s+2, \ldots, s+t\}$ define multiset $Y_j = \{c_j b_{1'}, c_j b_{2'}, \ldots, c_j b_{n'}, c_i\}$ where for $1 \leq u \leq n$ and $n \geq k+s$:

$c_j b_{u'} = \begin{cases} 1, \\ m, \end{cases}$ 0
if $1 \le j \le u$;
0 0 if $(m-2)n + m + u - 1 \le j \le (m-1)n +$
$m + u - 1$

and

$c_j = \begin{cases} 1, \\ m, \end{cases}$ if $1 \le j \le n + 1$; if $(m-1)n + m \le$
$j \le mn + m.$
0 0

For $n < k+s$, let $k+s-n = p + z(n+1) - y - x$ where $p \in \{1,2,...,n+1\}$, $z \ge$
1, $y \in \{0,1,...,n+1\}$, $x \in \{1,2,...,n\}$ and
z, if $1 \le j - s \le x$;

$z + 1$, if $x + 1 \le j - s \le x + y$; $c_j b_p 0 = z + 2$,

if $x + y + 1 \le j - s \le x + y + n + 1$;

$z + o$, if $x + y + (o-2)n + o - 1 \le j - s \le x + y + (o-1)n + o - 1$

where $c_j b_{n+1} 0$ defined as c_j.

Next, define

$A_l = \{a_i b(l)(i) | 1 \le i \le s\}$

$A_{k+1} = \{a_i | 1 \le i \le s\}$

$A_{k+u0+1} = \{c_j b_u 0 | 1 \le j \le t\}$ A_{k+n+2}

$= \{c_j | 1 \le j \le t\}$.

$k+1$ $k+n+2$
0 where $1 \le l \le k$ and $1 \le u \le n$. We have $\bigcup A_i = X$, $\bigcup$
$A_i = Y$, and
$i=1$ $i=k+2$

s $s+t$

$\bigcup X_i = X$, $\bigcup Y_i = Y$. Besides that, for each $i \in \{1,2,...,s\}$, $|Y_i| = k + 1$ and
$i=1$ $i=s+1$ for each $j \in \{s+1, s+2,...,s+t\}$,
$|Y_j| = n + 1$.

Case 1: $n \ge k + s$

Consider a_i and a_{i+1} where $i \in \{1,2,...,s-1\}$. Let $a_i b_{(l)(i)} = m$, then

$$(m-2)k + m + l - 1 \le i \le (m-1)k + m + l - 1$$
$$(m-2)k + m + l \le i + 1 \le (m-1)k + m + l$$
$$(m-2)k + m + (l+1) - 1 \le i + 1 \le (m-1)k + m + (l+1) - 1.$$

So, $a_i b_{(l)}(i) = m = a_{i+1} b_{(l+1)}(i+1)$ and

$$\begin{aligned} \Sigma X_{i+1} &= a_{i+1} + a_{i+1} b(k)(i+1) + ... + a_{i+1} b(1)(i+1) \\ &= a_i b(k)(i) + a_i b(k-1)(i) + ... + a_i b(1)(i)) + a_{i+1} b(1)(i+1) \\ &= \Sigma X_i - a_i + a_{i+1} b_{(1)(}i+1). \end{aligned} \quad (1)$$

Let $a_i = m$, then

$(m-1)k+m \le i \le mk+m$

$$((m+1)-2)k+m \le i \le ((m+1)-1)k+m$$

$$((m+1)-2)k+m+1 \le i+1 \le ((m+1)-1)k+m+1.$$

So $a_{i+1}b_{(1)(i+1)} = m+1$. From (2) we have

$$\Sigma X_{i+1} = \Sigma X_i + 1.$$

By using a similar way, we have $\Sigma Y_{j+1} = \Sigma Y_j + 1$, $j \in \{s+1, s+2, ..., s+t-1\}$. Hence, if $n \ge k+s$, we obtained $\Sigma X_k \neq \Sigma X_l$ for $k \neq l$, $k,l \in \{1,2,...,s\}$ and $\Sigma Y_p \neq \Sigma Y_q$ for $p \neq q$, $p,q \in \{s+1, s+2, ..., s+t\}$.

Case 2: $n < k+s$

Consider c_j and c_{j+1} where $j \in \{s+1, s+2, ..., s+t-1\}$. Let $c_j b_u 0 = z+o$, then

$$x+y+(o-2)n+o-1 \le j \le x+y+(o-1)n+o-1 \; x+$$

$$y+(o-2)n+o \le j+1 \le x+y+(o-1)n+o$$

0

Since $x+y = u+z(n+1)-k-s$,

$$0 \; x+y+(o-2)n+o = (u+1)+z(n+1)-k-s+(o-2)n+o-1.$$

So, $cjbu0 = z+o = cj+1bu+10$ and

$$\begin{aligned}
\Sigma Yj{+}1 &= c_j{+}1 + c_j{+}1^{b}{}_{n}0 + ... + c_j{+}1^{b}{}_{1}0 \\
&= cjbn0 + cjbn{-}10 + ... + cjb10 + cj{+}1b10 \\
&= \Sigma Yj - cj + cj{+}1b10. \qquad (2)
\end{aligned}$$

Let $c_j = z+o$, then $x+y+(o-2)n+o-1 \le j \le x+y+(o-1)n$

$+o-1$

$$x+y+(o-2)n+o \le j+1 \le x+y+(o-1)n+o.$$

Since $cj = cjbn{+}10$,

$$x+y+(o-2)n+o = n+1+z(n+1)-k-s+(o-2)n+o$$

$$= 1+z(n+1)-k-s+((o+1)-2)n+(o+1)-1.$$

So $c_{j+1}b_1 0 = z+o+1$. From (2) we have

$$\Sigma Y_{j+1} = \Sigma Y_j + 1.$$

Hence, if $n < k+s$, we obtained $\Sigma X_k \neq \Sigma X_l$ for $k \neq l$, $k,l \in \{1,2,...,s\}$ and $\Sigma Y_p \neq \Sigma Y_q$ for $p \neq q$, $p,q \in \{s+1, s+2, ..., s+t\}$.

So X and Y are anti-balanced.

Theorem 2.2. *Let $n,s,$ and t be integers at least 2 and $k \in \{1,2,...,n-1\}$. Let $G = S(A,B) \circ H$ with $H = tK_1$, $|A| = s$, and $|B| = n$. If $deg(a) = k$ for all $a \in A$, then*

$$tvs(G) = \begin{cases} \lceil \frac{k+s}{k+1} \rceil, & n \ge \frac{t(k+1)}{s} \\ \lceil \frac{n+t}{n+1} \rceil, & n \ge k+s \text{ and } n < \frac{t(k+1)}{s}; \\ \lceil \frac{k+s+t}{n+1} \rceil, & n < k+s \text{ and } n < \frac{t(k+1)}{s}. \end{cases}$$

Proof. Let φ be an optimal total k-labeling with respect to the $tvs(G)$. To get the smallest k such that the weight of every vertices $v \in V(G)$ are distinct then the weight of the smallest sequence should started from vertices $a \in A$ whose degree are k, $1 \le k \le n-1$ with all edges joining that vertices to the vertices $b \in B$. Next, give label for the vertices $c \in V(H)$ whose degree are n with all edges joining that vertices to the vertices $b \in B$.

- Since the degree of every vertex in A is k, the smallest weight of all vertices in A is at least $k+1$ and the largest weight of all vertices in A is at least $k+s$.
- Since the degree of every vertex in C is n, the smallest weight of all vertices in H is at least
$$\begin{cases} n+1, & n \ge k+s \\ k+s+1, & n < k+s \end{cases};$$
and the largest weight of all vertices in H is at least
$$\begin{cases} n+t, & n \ge k+s \\ k+s+t, & n < k+s \end{cases}.$$

Thereby, the largest label of G is at least
$$\begin{cases} max\{\lceil \frac{k+s}{k+1} \rceil, \lceil \frac{n+t}{n+1} \rceil\}, & n \ge k+s \\ max\{\lceil \frac{k+s}{k+1} \rceil, \lceil \frac{k+s+t}{n+1} \rceil\}, & n < k+s \end{cases};$$

where
$$max\{\lceil \frac{k+s}{k+1} \rceil, \lceil \frac{n+t}{n+1} \rceil\} = \begin{cases} \lceil \frac{k+s}{k+1} \rceil, & n \ge \frac{t(k+1)}{s} \\ \lceil \frac{n+t}{n+1} \rceil, & n < \frac{t(k+1)}{s} \end{cases};$$

and
$$max\{\lceil \frac{k+s}{k+1} \rceil, \lceil \frac{k+s+t}{n+1} \rceil\} = \begin{cases} \lceil \frac{k+s}{k+1} \rceil, & n \ge \frac{t(k+1)}{s} \\ \lceil \frac{k+s+t}{n+1} \rceil, & n < \frac{t(k+1)}{s} \end{cases}.$$

Therefore,
$$tvs(G) \ge \begin{cases} \lceil \frac{k+s}{k+1} \rceil, & n \ge \frac{t(k+1)}{s} \\ \lceil \frac{n+t}{n+1} \rceil, & n \ge k+s \text{ and } n < \frac{t(k+1)}{s}; \\ \lceil \frac{k+s+t}{n+1} \rceil, & n < k+s \text{ and } n < \frac{t(k+1)}{s}. \end{cases}$$

Next, we will show that
$$tvs(G) \le \begin{cases} \lceil \frac{k+s}{k+1} \rceil, & n \ge \frac{t(k+1)}{s} \\ \lceil \frac{n+t}{n+1} \rceil, & n \ge k+s \text{ and } n < \frac{t(k+1)}{s}; \\ \lceil \frac{k+s+t}{n+1} \rceil, & n < k+s \text{ and } n < \frac{t(k+1)}{s}. \end{cases}$$

For each vertex a_i, $i \in \{1,2,...,s\}$, define $\{b_{(1)(i)}, b_{(2)(i)}, ..., b_{(k)(i)}\}$ as its neighbor set and indexing them based on index in $b_j \in B$, $j \in \{1,2,...,n\}$. Give label edges $a_i b_{(l)(i)}$, $l \in \{1,2,...,k\}$, and vertices a_i:

$$\lambda(a_i b_{(l)(i)}) = \begin{cases} 1, & 1 \le i \le l; \\ m, & (m-2)k \quad l \quad \le \quad \le \quad k \quad l + m + -1 \quad i \end{cases}$$

$(m-1) + m + -1,$

$$\lambda(a_i) = \begin{cases} 1, & 1 \le i \le k+1; \\ m, & (m-1)k + m \le i \le mk + m, \end{cases}$$

and $wt(ap) = \lambda(ap) + \lambda(apb(k)(p)) + \lambda(apb(k-1)(p)) + ... + \lambda(apb(1)(p))$.

For vertices in B, define $f(b_j)$ to be the sum of the labels of all edges joining b_j to the vertex $a_i \in A$. Denote the vertices of B as $b_10, b_20, ..., b_n0$, indexing them in non increasing order their values under f. Next, give label the vertices in H and edges c_ob_j0, $o \in \{s+1, s+2, ..., s+t\}$, by using the similar way as label that given for the vertices in A and edges $a_ib_{(l)(i)}$ with the next smallest number. Then give label vertices $b_j0 \in B$ with all edges joining that vertices to $b_q0 \in B \setminus \{b_j0\}$ by using the similar way as label that given to vertices A dan $V(H)$ with all edges incident with that vertices. Since $f(b_n0)$ is the smallest one, we give label vertices $b_j0 \in B$ start from b_n0.

From Lemma 2.1 we have

- for each $a_i, a_{i+1} \in A$, $wt(a_{i+1}) = wt(a_i) + 1$; • for each $c_o, c_{o+1} \in V(H)$, $wt(c_{o+1}) = wt(c_o) + 1$; and
- $wt(a_i) < wt(c_o)$.

Let b_i0 and b_j0 be vertices of B with $i < j$; by definition, $f(b_i0) \ge f(b_j0)$. Since $deg(c_o) = n < n-1+t \le deg(b_n0)$ and the way to give label of each vertex in B is similar as in A and $V(H)$ then we obtained $wt(c_t) < wt(b_n0)$ and $wt(b_j0) < wt(b_i0)$. Furthermore, we have $wt(a_i) < wt(c_o) < wt(b_j0)$.

Therefore,

$$tvs(G) \le \begin{cases} \lceil \frac{k+s}{k+1} \rceil, & n \ge \frac{t(k+1)}{s}; \\ \lceil \frac{n+t}{n+1} \rceil, & n \ge k+s \text{ and } n < \frac{t(k+1)}{s}; \\ \lceil \frac{k+s+t}{n+1} \rceil, & n < k+s \text{ and } n < \frac{t(k+1)}{s}. \end{cases}$$

Hence,

$$tvs(G) = \begin{cases} \lceil \frac{k+s}{k+1} \rceil, & n \ge \frac{t(k+1)}{s}; \\ \lceil \frac{n+t}{n+1} \rceil, & n \ge k+s \text{ and } n < \frac{t(k+1)}{s}; \\ \lceil \frac{k+s+t}{n+1} \rceil, & n < k+s \text{ and } n < \frac{t(k+1)}{s}. \end{cases}$$

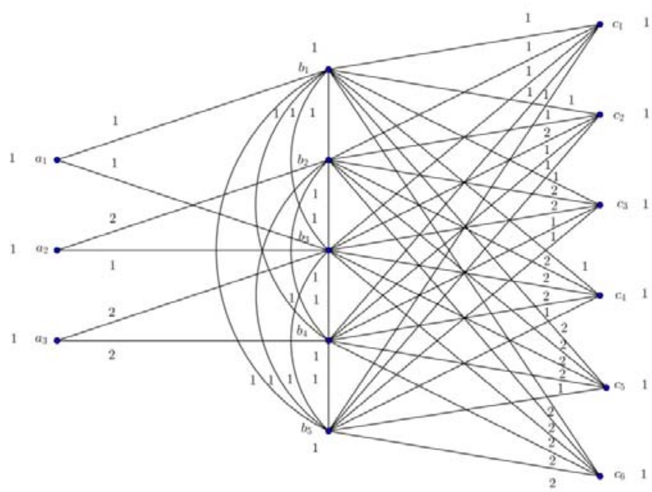

Figure 2. A vertex-irregular total *2*-labeling for $G = S(A,B) \circ$

$6K_1$ with $|A| = 3$, $|B| = 5$ and $deg(a) = 2$ for all $a \in A$

From Figure 2 above we can see that two multisets $X = \{1^6,2^3\}$ and $Y = \{1^{21},2^{15}\}$ are anti-balanced since we have following multisets from X and Y :

- $X_1 = \{1,1,1\}$; • $X_2 = \{1,1,2\}$;
- $X_3 = \{1,2,2\}$;
- $Y_4 = \{1,1,1,1,1,1\}$; • $Y_5 = \{1,1,1,1,1,2\}$; • $Y_6 = \{1,1,1,1,2,2\}$; • $Y_7 = \{1,1,1,2,2,2\}$; • $Y_8 = \{1,1,2,2,2,2\}$; • $Y_9 = \{1,2,2,2,2,2\}$.

References

[1] O. Al-Mushayt, A. Arshad, M.K. Siddiqui, Total vertex irregularity strength of convex polytope graphs, *Acta Math. Univ. Cominianae* **82** (2013) 29-37.

[2] M. Baˇca, S. Jendról, M. Miller, J. Ryan, On irregular total labellings, *Discrete Math.*, **307** (2007) 1378-1388.

[3] Nurdin, E.T. Baskoro, A.N.M. Salman, N.N. Gaos, On the total vertex-irregularity strength of trees, *Discrete Math.* **310** (2010) 3043-3048.

[4] Nurdin, E.T. Baskoro, A.N.M. Salman, N.N. Gaos, On total vertex-irregular labellings for several types of trees, *Util. Math.* **83** (2010), 277-290.

[5] Nurdin, E.T. Baskoro, A.N.M. Salman, N.N. Gaos, On the total vertex-irregular strength of a disjoint union of t copies of a path, *J. Combin. Math. Combin. Comput.* **71** (2009) 227-233.

[6] C. Tong, X. Lin, Y. Yang, L. Wang, Irregular total labellings of $C_m \Box C_n$, *Util. Math.* **81** (2010) 3-13.

[7] R. Tyshkevich, Decomposition of graphical sequences and unigraphs, *Discrete Math.*, **220** (2000) 201-238.

[8] K. Wijaya, Slamin, Surahmat, S. Jendroľ, Total vertex-irregular labellings of complete bipartite graphs, *J. Combin. Math. Combin. Comput.* **55** (2005) 129-136.

THE ODD HARMONIOUS LABELING OF kC$_n$-SNAKE GRAPHS FOR SPESIFIC VALUES OF n, THAT IS, FOR $n = 4$ AND $n = 8$

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Abstract. A graph $G = (V(G), E(G))$is called (p, q)-graph if it has$p$ vertices and q edges. A (p, q)-graph G is said to be an odd harmonious if there exists an injection $f: V(G) \to [0, 2q - 1]$ such that the induced $f^*(uv) = f(u) + f(v)$ is a bijection from E onto $[1, 2q - 1]_2 = \{1,3,5, \ldots, 2q - 1\}$ and f is said to be an odd harmonious labeling of G. A graph is called kC$_n$-snake graphs with $k \geq 1$ is a connected graph with k blocks whose block-cutpoint graph is a path and each of the k blocks is isomorphic to C$_n$.In this paper, we constructthe odd harmonious labeling on kC$_n$-snake graph for several values of n. Moreover, we also give odd harmonious labeling construction for a class of graph which is variation of kC$_n$-snake graph.

Key words and Phrases: Odd harmonious labeling,kC$_n$-snake graph, Snake graph.

1. Introduction

In this paper we consider simple, finite and undirected graph. A graph $G = (V(G), E(G))$is called (p, q)-graph if it has$p$ vertices and q edges. There are several types of graph labeling, some of them have been motivated by practical problems, but most of its come from the coriousities of the beautiful of the art of graph labelings. We refer to Gallian [1] for a dynamic survey of various graph labeling problems along with extensive bibliography. Let Z be the ring of integer, and Z_n be the residue ring of integers modulo n, the simbol $[a, b]$ is defined by $\{x | x \in Z, a \leq x \leq b\}$, and $[a, b]_k$ is defined by $\{x | x \in Z, a \leq x \leq b, x \equiv a (\text{mod } k)\}$. If S is a set, $f(S)$ denotes the set $\{f(x) | x \in S\}$ [3]. Graham and Sloane [2] intoduced harmonious labeling and defined as follows :

Definition 1.1 [2] *A (p, q)-graph G is said to be harmonious if there exists an injection $f: V(G) \to Z_q$ such that the induced mapping $f^*(uv) = (f(u) +$*

$f(v))(mod q)$ *is a bijection from* E *onto* Z_q *and* f *is said to be harmonious labeling of* G.

Liang and Bai [3] intoduced odd harmonious labeling and defined as follows :

Definition 1.2 [3] *A* (p, q)*-graph* G *is said to be odd harmonious if there exists an injection* $f: V(G) \to [0, 2q-1]$, *such that the induced mapping* $f^*(uv) = f(u) + f(v)$ *is a bijection from* E *onto* $[1, 2q-1]_2 = \{1,3,5, \dots, 2q-1\}$ *and* f *is said to be a odd harmonious labeling of* G.

Liang and Bai [3] have obtained the necessary conditions for the existence of odd harmonious labeling of graph. Futhermore, they proved if G is an odd harmonious graph, then G is a bipartite graph and if a (p, q)-graph G is odd harmonious, then is the number of vertices is bound by $\sqrt{q} \leq p \leq 2q-1$. The maximal label of all vertices in an odd harmonious graph G is at most $2q - \delta(G)$, where $\delta(G)$ is the minimum degree of the vertices of G. Let $(d_1, d_2, \dots, d_p)$ be a degree sequence of (p, q)-graph G, if the graph G is odd harmonious, then the equation $\sum_{i=1}^{p} d_i x_i = q^2$ has non-negative integer solutions $(x_1, x_2, \dots, x_p)$ satisfying $x_i \neq x_j$ if $i \neq j$ and $x_i \leq 2q - \delta(G)$ for $i \in [1, p]$. Let $(d_1, d_2, \dots, d_p)$ be a degree sequence of (p, q)-graph G then the necessary conditions for a graph G to be odd harmonious is $gcd(d_1, d_2, \dots, d_p) | q^2$. If all trees are odd harmonious, then the equation $\sum_{i=1}^{p} d_i x_i = (p-1)^2$ has distinct non-negative integer solutions for any integer p satisfying $\sum_{i=1}^{p} d_i = 2(p-1)$ and $d_i \geq 0, i \in [1, p]$.

In the same paper Liang and Bai [3] proved that cycle C_n is odd harmonious if and only if $n \equiv 0 (\text{mod } 4)$, a complete graph K_n is odd harmonious if and only if $n = 2$, a complete k-partite graph $K(n_1, n_2, \dots, n_k)$ is odd harmonious if and only if $k = 2$, a windmill graph K_n^t is odd harmonious if and only if $n = 2$. They also proved that the graph $\vee_{i=1}^{n} G_i$, the graph $G(+r_1, +r_2, \dots, +r_p)$, the graph $\overline{K}_m +_0 P_n +_e \overline{K}_t$, the graph $G \cup (X + \cup_{k=1}^{n} Y_k)$, some trees and the product graph $P_m \times P_n$ etc are odd harmonious graph.

Vaidya and Shah [6] proved that the shadow graphs of path P_n and star $K_{1,n}$ are odd harmonious. Futhermore, they proved that the split graphs of path P_n and star $K_{1,n}$ admited odd harmonious labeling.

Saputri [5] proved that ladder graphs L_n for $n \geq 2$, dumbbell graphs $D_{n,n,2}$ for $n \equiv 0 (\text{mod } 4), n \geq 4$, palm graphs $C_n - B_{m,k}$ for $n \equiv 0 (\text{mod } 4)$, $n \geq 4$, $m \geq 3$, $k \geq 1$, generalized prism graphs $C_n \times P_m$ for $n \equiv 0 (\text{mod } 4)$, $m \geq 2$ and sun graphs $C_n \odot K_1, n \equiv 0 (\text{mod } 4), n \geq 2$ admited odd harmonious labeling.

Rismayanti [4] proved that corona graphs $C_n \odot \overline{K}_r$ for $n \equiv 0 (\text{mod } 4)$, $r \geq 2$, sun graphs $C_n \odot \overline{K}_1$ for $n \equiv 0 (\text{mod } 4)$, hairy cycle graphs $HC(n; r_i)$ for $n \equiv 0 (\text{mod } 4), 1 \leq i \leq n-1, r_i \geq 2$, $r_n \geq 1$, cycle shadow graphs $D_2(C_n)$ for $n \equiv 0 (\text{mod } 4)$ and generalized of cycle shadow graphs $D_m(C_n)$ for $n \equiv 0 (\text{mod } 4)$, $m \geq 3$ admited odd harmonious labeling.

2. Main Results

Definition 2.1 [1]kC_4*-snake graphs with* $k \geq 1$ *is a connected graph with* k *blocks whose block-cutpoint graph is a path and each of the* k *blocks is isomorphic to* C_4.

The vertex notation and construction of kC_4-snake graphs is shown in Figure 2.

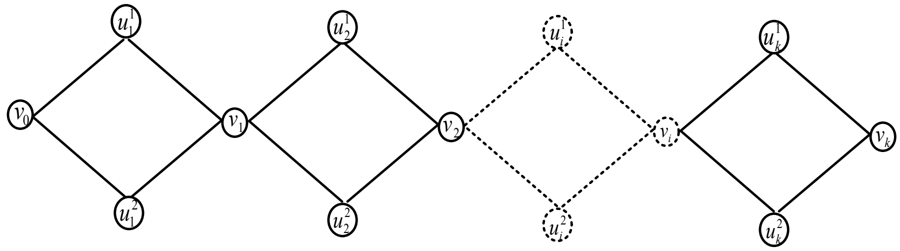

Figure 1 : kC_4-snake graphs

Definition 2.2 [1]kC_8*-snake graphs with* $k \geq 1$ *is a connected graph with* k *blocks whose block-cutpoint graph is a path and each of the* k *blocks is isomorphic to* C_8.(Gallian,2012).

The vertex notation and construction of kC_8-snake graphs is shown in Figure 3.

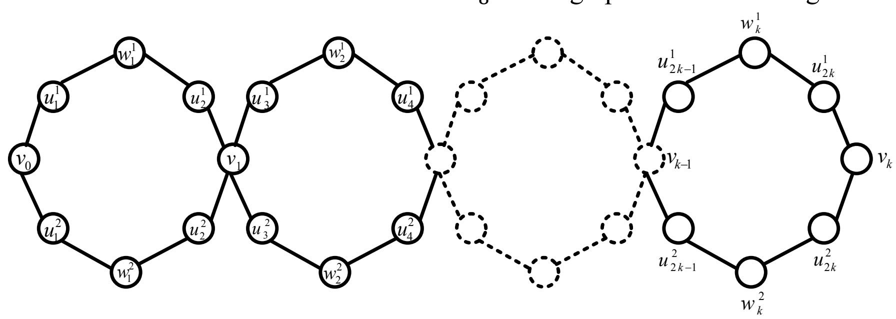

Figure 2 : kC_8-snake graphs

Definition 2.3[1]*Bracelet of* $C_4^{+(1,k)}$ *graph is a connected graph which consists of* k *block whose block-cutpoint graph is a path and each of the* k *blocks is isomorphic to* $C_4^{+(1,k)}$.

The vertex notation and construction of Bracelet of $C_4^{+(1,k)}$ graph is shown Figure 3.

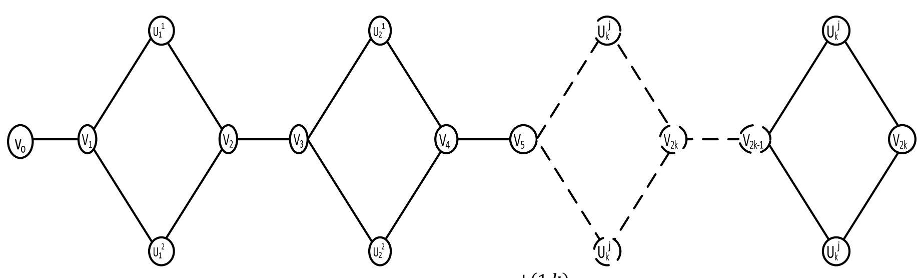

Figure 3 : Bracelet of $C_4^{+(1,k)}$ graphs

Theorem 2.1kC_4*-snake graphs with* $k \geq 1$ *is an odd harmonious graph.*

Proof. Let G be the graph kC_4 with ≥ 1 . The vertex set and edge set of kC_4 are defined as follows $V(kC_4) = \{v_0\} \cup \{u_i{}^j | 1 \leq i \leq k, j = 1,2\} \cup \{v_i | 1 \leq i \leq k\}$and

$E(kC_4) = \{v_i u_{(i+1)}{}^j | 0 \leq i \leq k-1, j = 1,2\} \cup \{u_i{}^j v_i | 1 \leq i \leq k, j = 1,2\}$

then $p = |V(kC_4)| = 3k + 1$ and $q = |E(kC_4)| = 4k$

Define the vertex labels $f: V(kC_4) \to \{0,1,2, \ldots, 8k-1\}$ as follows

$f(v_0) = 0$

$f(u_i{}^j) = 4i + 2j - 5, 1 \leq i \leq k, j = 1,2$

$f(v_i) = 4i, 1 \leq i \leq k$

The labeling f will induced mapping define $f^*: E(kC_4) \to \{1,3,5,7, \ldots, 8k-1\}$ which is defined by $f^*(uv) = f(u) + f(v)$. Thus we have the edge labels as follows

$f^*(v_i u_{(i+1)}{}^j) = f(v_i) + f(u_{(i+1)}{}^j) = 8i + 2j - 1, 0 \leq i \leq k-1, j = 1,2$

$f^*(u_i{}^j v_i) = f(u_i{}^j) + f(v_i) = 8i + 2j - 5, 1 \leq i \leq k, j = 1,2$

It is not difficults to show that the mapping f is an injective mapping and the mapping f^* admits a bijective labeling. Hence kC_4-snake graphs with $k \geq 1$ is an odd harmonious graph. □

Since the labeling for odd k and even k are different. In Figure 6 and 7, we give examples of odd harmonious labeling for $3C_4$and $4C_4$-snake graphs.

Theorem 2.2kC_8*-snake graphs with* $k \geq 1$ *is an odd harmonious graph.*

Proof. Let G be the graph kC_8 with $k \geq 1$. The vertex set and edge set of kC_8 are defined as follows $V(kC_8) = \{v_0\} \cup \{u_i{}^j | 1 \leq i \leq 2k, j = 1,2\} \cup \{w_i^j | 1 \leq i \leq k, j = 1,2\} \cup \{v_i | 1 \leq i \leq k\}$ and

$E(kC_8) = \{v_i u_{(2i+1)}{}^j | 0 \leq i \leq k-1, j = 1,2\} \cup \{v_i u_{(2i)}{}^j | 1 \leq i \leq k, j = 1,2\} \cup$

$\{w_i^j u_{2i}^j | 1 \leq i \leq k, j = 1,2\} \cup \{w_i^j u_{2i-1}^j | 1 \leq i \leq k, j = 1,2\}$

then $p = |V(kC_8)| = 16k + 1$ and $q = |E(kC_8)| = 8k$

Define the vertex labels $f: V(kC_8) \to \{0,1,2, \ldots, 8k-1\}$ as follows

$f(v_0) = 0$

$f(u_i{}^j) = 4i + 2j - 5, \quad 1 \leq i \leq 2k, \; j = 1,2$

$f(w_i{}^j) = 8i - 4j + 2, \quad 1 \leq i \leq k, \quad j = 1,2$

$f(v_i) = 8i, \quad 1 \leq i \leq k$

The labeling f will induced mapping define $f^*: E(kC_8) \to \{1,3,5,7, \ldots, 8k-1\}$ which is defined by $f^*(uv) = f(u) + f(v)$. Thus we have the edge labels as follows

$f^*(v_i u_{(2i+1)}{}^j) = f(v_i) + f(u_{(2i+1)}{}^j) = 16i + 2j - 1, \quad 0 \leq i \leq k-1, \; j = 1,2$

$f^*(v_i u_{(2i)}{}^j) = f(v_i) + f(u_{(2i)}{}^j) = 16i + 2j - 5, \quad 1 \leq i \leq k, \quad j = 1,2$

$f^*(w_i{}^j u_{(2i)}{}^j) = f(w_i{}^j) + f(u_{(2i)}{}^j) = 16i - 2j - 3, 1 \leq i \leq k, j = 1,2$

$f^*(w_i{}^j u_{(2i-1)}{}^j) = f(w_i{}^j) + f(u_{(2i-1)}{}^j) = 16i - 2j - 7, 1 \leq 1,2k, j = 1,2$

It is not difficults to show that the mapping f is an injective mapping and the mapping f^* admits a bijective labeling. Hence kC_8-snake graphs with $k \geq 1$ is an odd harmonious graph. □

Theorem 2.3*bracelet of* $C_4^{+(1,k)}$*graphs with* $k \geq 1$ *is an odd harmonious graph.*

Proof. Let G be the bracelet of $C_4^{+(1,k)}$graphs with $k \geq 1$. The vertex set and edge set of bracelet of $C_4^{+(1,k)}$graphsare defined as follows $V\left(C_4^{+(1,k)}\right) = \{v_i | 0 \leq i \leq 2k, i\ genap\} \cup \{v_i | 1 \leq i \leq 2k-1, i\ ganjil\} \cup \{u_i^j | i = 1,2,3,\dots,k\ , j = 1,2\}$ and

$$E\left(C_4^{+(1,k)}\right) = \{v_i v_{i+1} | i\ \ genap, 0 \leq i \leq 2k\} \cup \left\{v_i {u_{\frac{1}{2}i}}^j \middle| i\ genap, 1 \leq i \leq 2k, j = 1,2\right\} \cup \left\{v_i {u_{\frac{i+1}{2}}}^j \middle| i\ ganjil, 1 \leq i \leq 2k-1, j = 1,2\right\}$$

then $p = \left|V\left(C_4^{+(1,k)}\right)\right| = 5k - 1$ and $q = \left|E\left(C_4^{+(1,k)}\right)\right| = 5k$

Define the vertex labels $f: V(C_4^{+(1,k)}) \rightarrow \{0,1,2,\dots,10k-1\}$ as follows

$$f(v_i) = \begin{cases} \frac{5i}{2} & ,\ 0 \leq i \leq 2k,\ \ i\ \ genap \\ \frac{5i-3}{2} & ,\ 1 \leq i \leq 2k-1,\ \ i\ \ ganjil \end{cases}$$

$$f(u_i^j) = 5i + 2j - 5 \qquad , i = 1,2,3,\dots,k \quad \text{j=1,2}$$

The labeling f will induced mapping define $f^*: E(C_4^{+(1,k)}) \rightarrow \{1,3,5,7,\dots,10k-1\}$ which is defined by $f^*(uv) = f(u) + f(v)$. Thus we have the edge labels as follows

$$f^*(v_i v_{i+1}) = f(v_i) + f(v_{i+1}) = 5i + 1,\ i\ \ genap, 0 \leq i \leq 2k$$

$$f^*\left(v_i {u_{\frac{1}{2}i}}^j\right) = f(v_i) + f\left({u_{\frac{1}{2}i}}^j\right) = 5i + 2j - 5,\ i\ genap, 0 \leq i \leq 2k\ , j = 1,2$$

$$f^*\left(v_i {u_{\frac{i+1}{2}}}^j\right) = f(v_i) + f\left({u_{\frac{i+1}{2}}}^j\right) = 5i + 2j - 4,\ i\ ganjil, 1 \leq i \leq 2k-1\ , j = 1,2$$

It is not difficults to show that the mapping f is an injective mapping and the mapping f^* admits a bijective labeling. Hence bracelet of $C_4^{+(1,k)}$graphswith $k \geq 1$ is an odd harmonious graph. □

Example 2.4 Odd harmonious labeling for $4C_4$-snake graph is shown in Figure 5.

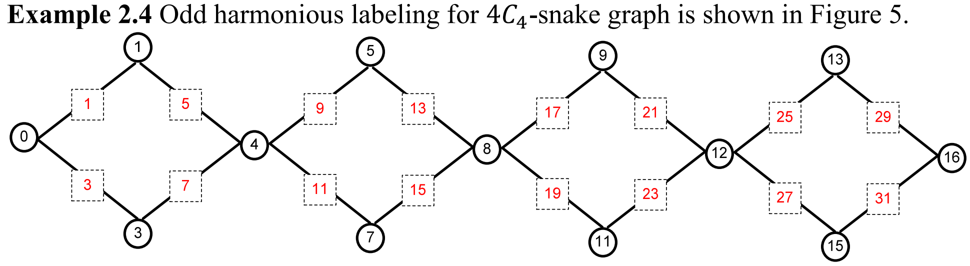

Figure 4 : $4C_4$-snake graphs and its odd harmonious labeling

Example 2.5 Odd harmonious labeling for $3C_8$-snake graph is shown in Figure 6

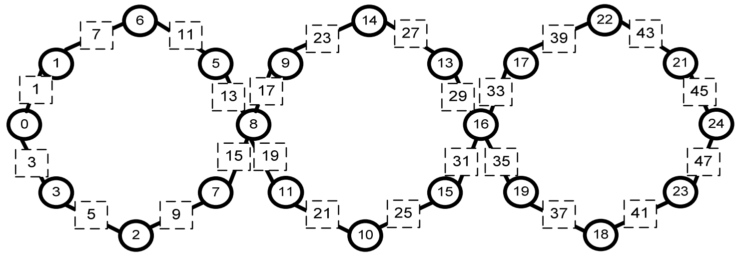

Figure 5 : $3C_8$-snake graphs and its odd harmonious labeling

Example 2.6 Odd harmonious labeling for bracelet of $C_4^{+(1,4)}$graphsis shown in Figure 6

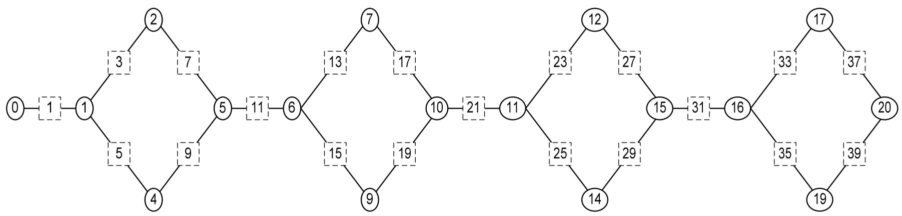

Figure 6 : bracelet of $C_4^{+(1,4)}$graphsand its odd harmonious labeling

3. Concluding Remarks

Acknowledgements. The authors are thankful to the anonymous referee for valuable comments and kind suggestions.

References

[1] Gallian, J. A., 2012, Dynamic Survey of Graph Labeling,*Electronic Journal of Combinatorics, 26.*

[2] Graham, R. L., Sloane, N. J. A., 1980, On Additive Bases and Harmonious Graphs, *SIAM J.Algebra. Disc.Math.*, Vol 1, No 4, 382-404.

[3] Liang, Z., Bai, Z., 2009, On The Odd Harmonious Graphs with Applications,J. *Appl. Math. Comput.,29,* 105-116.

[4] Rismayati., 2012, *Pelabelan Harmonis Ganjil pada Graf Hairy Cycle, Graf Shadow Lingkaran dan Graf Generalisasi Graf Shadow Lingkaran*, Tesis S2, Departemen Matematika., Universitas Indonesia., Depok.

[5] Saputri, G. A., 2013, The Odd Harmonious Labeling of Dumbbell and Generalized Prism Graphs, *AKCE Int. J. Graphs Comb.,* 10, No. 2(2013), pp. 221-228.

[6] Vaidya, S. K., Shah, N.H., 2011, Some New Odd Harmonious Graphs. *IJMSC*, Vol 1, No 1, 9-16.

CONSTRUCTION OF (a, d)-VERTEX-ANTIMAGIC TOTAL LABELINGS OF UNION OF TADPOLE GRAPHS

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Abstract. Let G be a graph with vertex set V and edge set E, where $|V|$ and $|E|$ be the number of vertices and edges of G. A bijection f from $V \cup E$ to the set $\{1, 2, \dots, |V| + |E|\}$ is an **(a, d)-vertex-antimagic total labeling** of G if the vertex weight set form an arithmetic progression with the initial term $a > 0$ and the common difference $d \geq 0$. In this paper, we give the construction of (a, d)-vertex-antimagic total labeling on disjoint union of m tadpole graphs, for $d = 1$.

Key words and Phrases : (a, d)-vertex-antimagic total labeling, tadpole graph.

1. Introduction

In this paper all graphs are finite, simple, and undirected. The graph G has vertex set V and edge set E. The number of vertices and edges on G are $|V|$ and $|E|$, respectively.

A **total labeling** of a graph G is a mapping f from the set of vertices and edges of G to a set of numbers (usually positive integers). Bača et al. introduced the notion of an (a, d)-vertex-antimagic total labeling [2]. The **(a, d)-vertex-antimagic total labeling** (VTAL) of a graph G is a bijective mapping from $V \cup E$ to the set $\{1, 2, \dots, |V| + |E|\}$ such that the set of weight of vertices, that is $\left\{f(v) + \sum_{u \in N(v)} f(uv) \mid v \in V\right\}$ with $N(v)$ is the set of all vertices adjacent to v, form an arithmetic progression $a, a + d, a + 2d, \dots, a + (|V| - 1)d$ where the initial term $a > 0$ and the different $d \geq 0$ are two fixed integers. A graph is called **(*a, d*)-vertex-antimagic total** if it admits an (a, d)-vertex-antimagic total labeling. If $d = 0$ then we call f as a vertex-magic total labeling. The concept of the vertex-magic total labeling was introduced by MacDougall et al. [5].

A **tadpole graph** $T_{p,q}$ is formed by joining an end point of a path with q vertices to a vertex of a cycle with p vertices, where the end point of path is one of the vertices of cycle. The **union of m tadpole graphs**, $\bigcup_{j=1}^{m} T_{p_j, q_j}$, consists of vertex set $V = \{v_i{}^j \mid 1 \le i \le q_j, 1 \le j \le m\} \cup \{u_i{}^j \mid 1 \le i \le p_j, 1 \le j \le m\}$ where $v_1{}^j = u_1{}^j$ ($v_1{}^j$ the end point of jth path is $u_1{}^j$ one of the vertices of jth cycle) for every j and the edge set $E = \{v_i{}^j v_{i+1}{}^j \mid 1 \le i \le q_j - 1, 1 \le j \le m\} \cup \{u_i{}^j u_{i+1}{}^j \mid 1 \le i \le p_j - 1, 1 \le j \le m\} \cup \{u_{p_j}{}^j u_1{}^j \mid 1 \le j \le m\}$. The number of vertices and edges on $\bigcup_{j=1}^{m} T_{p_j,q_j}$ is equal, that is $|V| = |E| = \sum_{j=1}^{m}(p_j + q_j - 1)$.

There are several results on (a,d)-vertex-antimagic total labeling of particular classes of graphs. For examples, Bača et al. [2] gave some basic properties of an (a,d)-vertex-antimagic total labeling. They also showed that paths and cycles have (a,d)-vertex-antimagic total labelings for a wide variety of a and d. Gray and MacDougall [4] showed that a tadpole has a vertex-magic total labeling (or $(a,0)$-vertex-antimagic total labeling). Agnesti, Silaban, and Sugeng [1] showed that a tadpole has (a,d)-vertex-antimagic total labeling for $d = 1,2,3$ and particular values of a. For the case of disjoint union of graphs, Parestu, Silaban, and Sugeng [6] proved that $S_{n_1} \cup S_{n_2} \cup \ldots \cup S_{n_t}$ is (a,d)-vertex-antimagic total for $d = 1,2,3,4$, and 6 and for a particular value of a. Further results on vertex-magic total labeling and (a,d)-vertex-antimagic total labeling can be found in Gallian [3].

For a graph G, if δ is the smallest degree of G, then the minimum possible weight of vertices is at least $1 + 2 + \cdots + (\delta + 1)$, consequently

$$a \ge \frac{(\delta+1)(\delta+2)}{2}. \qquad (1)$$

If Δ is the largest degree of G, then the maximum weight of vertices is less than the sum of the $\Delta + 1$ largest labels, that is $(|V| + |E| - \Delta) + (|V| + |E| - \Delta + 1) + \cdots + (|V| + |E|)$. Thus,

$$a + (|V| - 1)d \le \frac{(\Delta+1)(2(|V|+|E|)-\Delta))}{2}. \qquad (2)$$

Then we obtain the restriction of d as follows

$$d \le \frac{(\Delta+1)(2(|V|+|E|)-\Delta)-(\delta+1)(\delta+2)}{2(|V|-1)}. \qquad (3)$$

Since tadpole has $\delta = 1$ and $\Delta = 3$, then we have

$$d \le 7. \qquad (4)$$

In this paper we give an (a,d)-vertex-antimagic total labeling of $mT_{p,q}$ and $\bigcup_{j=1}^{m} T_{p_j,2}$ for $d = 1$.

2. Main Results

Theorem 1 gives the $(a,1)$-vertex-antimagic total labeling of m isomorphic copies tadpole $T_{p,q}$.

Theorem 1. *For $p \ge 3$ and $q \ge 2$, the tadpole $mT_{p,q}$ has a $((2p + 2q - 2)m + 2, 1)$-vertex-antimagic total labeling, where $m \ge 2$.*

PROOF. Label the vertices and the edges of the tadpole $mT_{p,q}$ as follows

- For $q = 2$:

$$f(u_i{}^j) = \begin{cases} 2pm & ,i = 1, j = 1 \\ (2p-1)m + j - 1 & ,i = 1, j \geq 2 \\ (2p-i+1)m - j + 1 & ,2 \leq i \leq p-1, j \geq 1 \\ 2pm + j & ,i = p, j \geq 1 \end{cases}. \quad (5)$$

$$f(v_i{}^j) = \begin{cases} (2p+2)m & ,i = q, j = 1 \\ (2p+1)m + j - 1 & ,i = q, j \geq 2 \end{cases}. \quad (6)$$

$$f(u_i{}^j u_{i+1}{}^j) = \begin{cases} j & ,i = 1, j \geq 1 \\ (i+1)m + j & ,2 \leq i \leq p-1, j \geq 1 \end{cases}. \quad (7)$$

$$f(u_p{}^j u_1{}^j) = 2m - j + 1 \quad ,j \geq 1. \quad (8)$$

$$f(v_i{}^j v_{i+1}{}^j) = \begin{cases} 3m & ,i = 1, j = 1 \\ 2m + j - 1 & ,i = 1, j \geq 2 \end{cases}. \quad (9)$$

- For $q \geq 3$:

$$f(u_i{}^j) = \begin{cases} (2p+2q-3)m - 2j + 1 & ,i = 1, j \geq 1 \\ (2p+2q-i-3)m - j + 1 & ,2 \leq i \leq p-1, j \geq 1. \\ (2p+2q-3)m - 2j + 2 & ,i = p, j \geq 1 \end{cases} \quad (10)$$

$$f(v_i{}^j) = \begin{cases} (p+q+i-2)m - j + 1 & ,2 \leq i \leq q-1, j \geq 1 \\ (2p+2q-3)m + j & ,i = q, j \geq 1 \end{cases}. \quad (11)$$

$$f(u_i{}^j u_{i+1}{}^j) = \begin{cases} j & ,i = 1, j \geq 1 \\ (i+1)m + j & ,2 \leq i \leq p-1, j \geq 1 \end{cases}. \quad (12)$$

$$f(u_p{}^j u_1{}^j) = 3m - j + 1 \quad ,j \geq 1. \quad (13)$$

$$f(v_i{}^j v_{i+1}{}^j) = \begin{cases} m + j & ,i = 1, j \geq 1 \\ (p+q-i)m + j & ,2 \leq i \leq q-1, j \geq 1 \end{cases}. \quad (14)$$

Under the labeling f we have

$$f(V) = \{(p+q-1)m + 1, (p+q-1)m + 2, \dots, (2p+2q-2)m\} \quad (15)$$

and

$$f(E) = \{1, 2, 3, \dots, (p+q-1)m\} \quad (16)$$

where $p \geq 3$ and $q \geq 2$. It means that the labeling f is a bijection from the set $V \cup E$ onto the set $\{1, 2, \dots, (2p+2q-2)m\}$.

Also, we have

$$w(u_i{}^j) = \begin{cases} f(u_i{}^j) + f(u_i{}^j u_{i+1}{}^j) + f(u_p{}^j u_i{}^j) + f(v_i{}^j v_{i+1}{}^j) & ,i = 1 \\ f(u_i{}^j) + f(u_i{}^j u_{i+1}{}^j) + f(u_{i-1}{}^j u_i{}^j) & ,2 \leq i \leq p_j - 1. \\ f(u_i{}^j) + f(u_i{}^j u_1{}^j) + f(u_{i-1}{}^j u_i{}^j) & ,i = p_j \end{cases} \quad (17)$$

$$w(v_i{}^j) = \begin{cases} f(v_i{}^j) + f(v_{i-1}{}^j v_i{}^j) & ,i = q_j \\ f(v_i{}^j) + f(v_i{}^j v_{i+1}{}^j) + f(v_{i-1}{}^j v_i{}^j) & ,2 \leq i \leq q_j - 1 \end{cases}. \quad (18)$$

By substituting equations (5)-(9) and (10)-(14) to (17)-(18) we get vertex weight set of $mT_{p,q}$ as follows.

$$\{w(u_i) \mid 1 \leq i \leq p\} \cup \{w(v_i) \mid 2 \leq i \leq q\}$$
$$= \{(2p+2q-2)m + 2, (2p+2q-2)m + 3, \dots, (3p+3)m + 1\} \quad (19)$$

for $q = 2$ and

$$\{w(u_i) \mid 1 \leq i \leq p\} \cup \{w(v_i) \mid 2 \leq i \leq q\}$$

$$= \{(2p + 2q - 2)m + 2, (2p + 2q - 2)m + 3, \dots, (3p + 3q - 3)m + 1\} \qquad (20)$$

for $q > 2$.

Since the vertex weight consists of consecutive integers starting from $(2p + 2q - 2)m$ with different 1, thus f is a $((2p + 2q - 2)m, 1)$-vertex-antimagic total labeling on tadpole $mT_{p,q}$. □

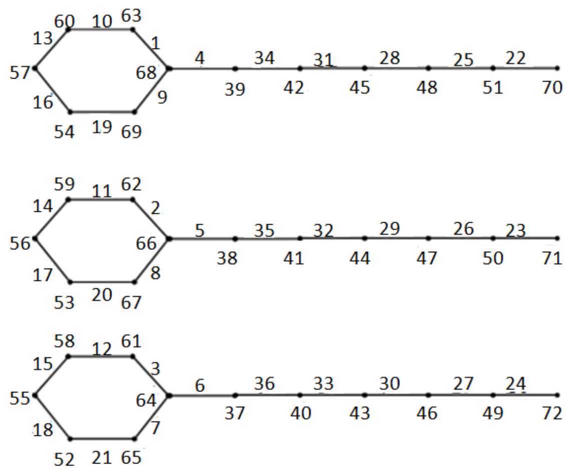

Figure 1. (74, 1)-VATL on $3T_{6,7}$

An example of a $(74, 1)$-vertex-antimagic total labeling of $3T_{6,7}$ can be seen at Figure 1.

For the union of non-isomorphic tadpoles, we get only for a special case, when $q_j = 2$ for all $j = 1, 2, \dots, m$, as given in Theorem 2.

Theorem 2. *For $p_j \geq 3, j = 1, 2, \dots, m$ the tadpole $\bigcup_{j=1}^{m} T_{p_j,2}$ has a $(2\sum_{j=1}^{m}(p_j + 1) + 2, 1)$-vertex-antimagic total labeling, where $m \geq 2$.*

Proof. Let $n_j = p_j + q_j - 1 = p_j + 1, j = 1, \dots, m$ and define

$$\alpha(a, b) = \begin{cases} a - b & , a > b \\ 0 & , \text{otherwise} \end{cases}.$$

For $j = 1, 2, \dots, m$, label the vertices and the edges of tadpole $\bigcup_{j=1}^{m} T_{p_j,2}$ as follows

$$f(u_i{}^j) = \begin{cases} 2\sum_{j=1}^{m} n_j - 2m - j + 1 & i = 1 \\ 2\sum_{j=1}^{m} n_j - m(i+1) - j + 1 + \sum_{j=1}^{m} \alpha(i, n_j) & i = 2, \dots, p_j - 1 \\ 2\sum_{j=1}^{m} n_j - m - j + 1 & i = p_j \end{cases} \qquad \begin{matrix}(21)\\(22)\end{matrix}$$

$$f(v_2{}^j) = 2\sum_{j=1}^{m} n_j - (m - j) \qquad (23)$$

$$f(u_i{}^j u_{i+1}{}^j) = \begin{cases} j & , i = 1 \\ j + m(i+1) - \sum_{j=1}^{m} \alpha(i, n_j) & , 2 \leq i \leq p - 1 \end{cases}. \qquad \begin{matrix}(24)\\(25)\end{matrix}$$

$f(u_p{}^j u_1{}^j) = j + m.$
$f(v_1{}^j v_2{}^j) = j + 2m.$

It can be checked that

$$f(V) = \{\textstyle\sum_{j=1}^{m} n_j + 1, \sum_{j=1}^{m} n_j + 2, \dots, 2\sum_{j=1}^{m} n_j\} \qquad (26)$$

and

$$f(E) = \{1, 2, 3, \dots, \textstyle\sum_{j=1}^{m} n_j\}. \qquad (27)$$

It means that the labeling f is a bijection from the set $V \cup E$ onto the set $\{1, 2, \dots, 2\sum_{j=1}^{m} n_j\}$.

By substituting equations (5)-(9) and (10)-(14) to (17)-(18) we get vertex weight set of $\bigcup_{j=1}^{m} T_{p_j,2}$ as follows.

$$\{w(u_i)\ |1 \le i \le p\} \cup \{w(v_2)\} = \{2\textstyle\sum_{j=1}^{m} n_j + 2, 2\sum_{j=1}^{m} n_j + 3, \dots, 3\sum_{j=1}^{m} n_j + 1\}. \qquad (28)$$

Since the vertex weight consists of consecutive integers starting from $2\sum_{j=1}^{m} n_j + 2 = 2\sum_{j=1}^{m}(p_j + 1) + 2$ with different 1, thus f is a $(2\sum_{j=1}^{m}(p_j + 1) + 2, 1)$-vertex-antimagic total labeling on tadpole $\bigcup_{j=1}^{m} T_{p_j,2}$. □

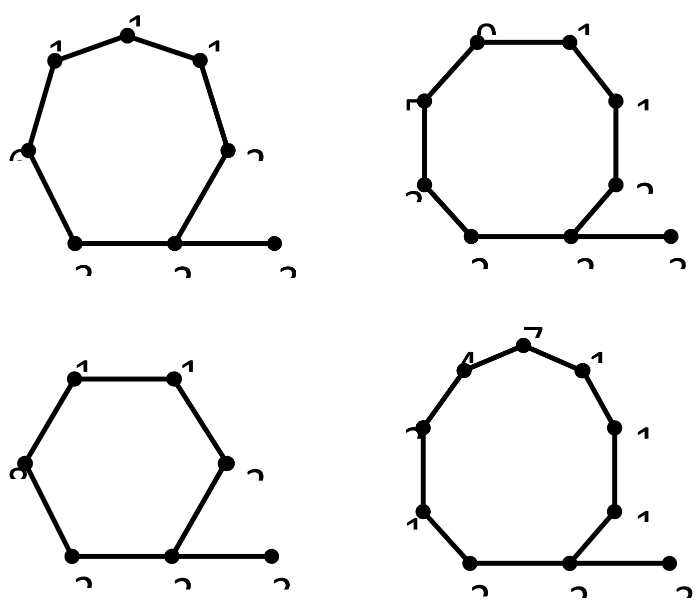

Figure 2. $(70, 1)$-VATL on $T_{7,2} \cup T_{8,2} \cup T_{6,2} \cup T_{9,2}$

An example of a $(70, 1)$-vertex-antimagic total labeling on $T_{7,2} \cup T_{8,2} \cup T_{6,2} \cup T_{9,2}$ can be seen at Figure 2.

3. Concluding Remarks

We conclude this paper with an open problem for further research as follows.

Open Problem. Find if there exists an (a, d)-vertex-antimagic total labeling on union of tadpole graphs for other values of d.

References

[1] Agnesti, P.T., Silaban, D.R., Sugeng, K.A., 2013, Construction of (a, d)-Vertex-Antimagic Total Labeling on Tadpole Graphs, *Proceeding The 1st Indonesian Student Conference on Science and Mathematics, to appear.*

[2] Bača, M. et al., 2003, Vertex-Antimagic Total Labelings of Graphs, *Discuss. Math. Graph Theory*, 23, 67–83.

[3] Gallian, J.A., 2013, A Dynamic Survey of Graph Labeling, *The Electronic Journal of Combinatorics 19*, #DS6.

[4] Gray, I.D., and MacDougall, J.A., 2008, Vertex-Magic Labelings: Mutations, *Australas. J. Combin*, v45, 189-206.

[5] MacDougall, A. et al., 2002, Vertex-Magic Total Labelings of Graphs, *Utilitas Mathematics*, 61, 68-76.

[6] Parestu, A., Silaban, D.R., Sugeng, K.A., 2008, Vertex Antimagic Total Labelings of Union of Suns, JCMCC, 71, 2009, 179-188.

SUPER ANTIMAGICNESS OF TRIANGULAR BOOK AND DIAMON LADDER GRAPHS

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Abstract. A graph G of order p and size q is called an (a,d)-*edge-antimagic total* if there exists a bijection $f : V(G) \cup E(G) \rightarrow \{1,2,\ldots,p+q\}$ such that the edgeweights, $w(uv) = f(u)+f(v)+f(uv), uv \in E(G)$, form an arithmetic sequence with first term a and common difference d. Such a graph G is called *super* if the smallest possible labels appear on the vertices. In this paper we study super (a,d)-edgeantimagic total properties of Triangular Book and Diamond Ladder graphs. The result shows that there are a super (a,d)-edge-antimagic total labeling of graph Bt_n and Dl_n, if $n \geq 1$ with $d \in \{0,1,2\}$.

Key Words: *(a,d)-edge-antimagic total labeling, super (a,d)-edge-antimagic total labeling, Triangular Book, Diamond Ladder.*

1. Introduction

Mathematics consists of several branches of science. Branch of current mathematics associated with a computer science is a graph theory. One of the interesting topics in graph theory is a graph labeling. Several applications of graph labeling can be found in [3]. There are various types of graph labeling, one is a super (a,d)edge antimagic total labeling (SEATL for short). Assigning a label on each vertex and each edge such that the edge-weights form an arithmetic sequence is considered to be an NP-complete problem.

By a *labeling* we mean any mapping that carries a set of graph elements onto a set of numbers, called *labels*. In this paper, we deal with labelings with domain the set of all vertices and edges. This type of labeling belongs to the class of *total* labelings. We define the *edge-weight* of an edge $uv \in E(G)$ under a total labeling to be the sum of the vertex labels corresponding to vertices u, v and edge label corresponding to edge uv.

These labelings, introduced by Simanjuntak *at al*. in [16], are natural extensions of the concept of magic valuation, studied by Kotzig and Rosa [14] (see also [2],[11],[12],[15],[18]), and the concept of super edge-magic labeling, defined by

Enomoto *et al.* in [10]. Many other researchers investigated different forms of antimagic graphs. For example, see Bodendiek and Walther [4] and [5], and Hartsfield and Ringel [13].

In this paper we investigate the existence of super (a,d)-edge-antimagic total labelings for connected graphs. Some constructions of super (a,d)-edge-antimagic total labelings for $m£_n$ and $m£_{i,j,k}$ have been shown by Dafik, Slamin, Fuad and Rahmad in [6] and super (a,d)-edge-antimagic total labelings for disjoint union of caterpillars have been described by Baˇca in [1]. Dafik *et al* also found some families of graph which admits super (a,d)-edge-antimagic total labelings, namely $mC_n, mP_n, mK_{\underbrace{n,n,\ldots,n}_{s}}$ and m caterpilars in [7, 8, 9].

All graphs in this paper are finite, undirected, and simple. For a graph G, $V(G)$ and $E(G)$ denote the vertex-set and the edge-set, respectively. A (p,q)-graph G is a graph such that $|V(G)| = p$ and $|E(G)| = q$. We will now concentrate on Triangular Book and Diamond Ladder graphs, denoted by Bt_n and Dl_n.

2. Three useful Lemmas

We start this section with a necessary condition for a graph to be a super (a,d)-edge-antimagic total, which will provide a least upper bound, for a feasible value d.

Lemma 1. [17] *If a (p,q)-graph is super (a,d)-edge-antimagic total then* $d \le \frac{2p+q-5}{q-1}$.

Proof. Assume that a (p,q)-graph has a super (a,d)-edge-antimagic total labeling $f: V(G) \cup E(G) \to \{1,2,...,p+q\}$ with the edge-weight set $W = \{w(uv) : uv \in E(G)\} = \{a, a+1, a+2, ..., a+(q-1)d\}$. The minimum possible edge weight in the labeling f is at least $1+2+p+1 = p+4$. Thus, $a \ge p+4$. On the other hand, the maximum possible edge weight is at most $(p-1)+p+(p+q) = 3p+q-1$. Hence $a+(q-1)d \le 3p+q-1$. From the last inequality, we obtain the desired upper bound for the difference d.2

The following lemma, proved by Figueroa-Centeno *et al.* in [11], proves a necessary and sufficient condition for a graph to be super edge-magic (super $(a,0)$edge-antimagic total).

Lemma 2. [11] *A (p,q)-graph G is super edge-magic if and only if there exists a bijective function $f: V(G) \to \{1,2,...,p\}$, such that the set $S = \{f(u)+f(v) : uv \in E(G)\}$ consists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with magic constant $a = p+q+s$, where $s = min(S)$ and $S = \{a-(p+1), a-(p+2), ..., a-(p+q)\}$.*

A similar lemma with Lemma 2, Baˇca, Lin, Miller and Simanjuntak, see [2], stated that a (p,q)-graph G is super $(a,0)$-edge-antimagic total if and only if there exists $(a-p-q,1)$-edge-antimagic vertex labeling. They extended the study with the following lemma.

Lemma 3. [2] *If (p,q)-graph G has an (a,d)-edge antimagic vertex labeling then G has a super$(a+p+q,d-1)$-edge antimagic total labeling and a super$(a+p+1,d+1)$edge antimagic total labeling.*

In this paper, we will use the last lemma to prove the existence of a super $(a,0)$-edge-antimagic total labeling and super $(a,2)$-edge-antimagic total labeling for triangular book Bt_n and diamond ladder Dl_n.

3. Triangular Book Bt_n

Triangular Book graph denoted byS Bt_n is a connected graph with vertex setS $V(Bt_n) = \{x_i; i = 1,2\}\ \{y_j, 1 \le j \le n\}$ and edge set $E(Bt_n) = \{x_1x_2\}\ \{x_iy_j; i = 1,2, 1 \le j \le n\}$. Thus $|V(Bt_n)| = p = n + 2$ and $|E(Bt_n)| = q = 2n + 1$.

If Triangular Book graph has a super (a,d)-edge-antimagic total labeling then it follows from Lemma 1 that the upper bound of d is $d \le 2$ or $d \in \{0,1,2\}$. The following theorem describes an $(a,1)$-edge-antimagic vertex labeling for Triangular Book graph.

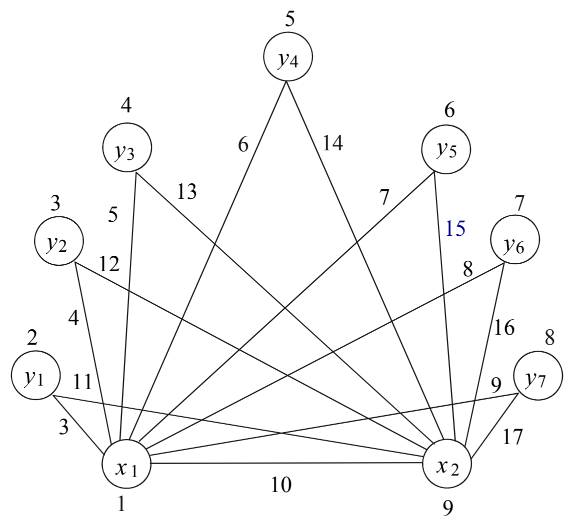

Figure 1. Example of (3,1)-edge antimagic vertex labeling Bt_7 with its edge weight

Theorem 1. *A triangular book Bt_n has an $(a,1)$-edge-antimagic vertex labeling if $n \ge 1$.*

Proof. Define the vertex labeling $\alpha_1 : V(Bt_n) \to \{1,2,...,n + 2\}$ in the following way:

$\alpha_1(x_i) = (i - 1)n + i$, for $i = 1, 2$
$\alpha_1(y_j) = j + 1$, for $1 \le j \le n$

The vertex labeling α_1 is a bijective function.

The edge-weights of Bt_n, under the labeling α_1, constitute the following sets

$$w_{\alpha 1}(x_1x_2) = n + 3$$

$$w_{\alpha 1}(x_iy_j) = n(i - 1) + j + (i + 1), \text{ for } i = 1, 2 \text{ dan } 1 \le j \le n$$

It is not difficult to see that the union of the set $w_{\alpha 1}$ equals to $\{3,4,5,...,n + 3,...,2n+3\}$ and consists of consecutive integers. Thus α_1 is a (3,1)-edge antimagic vertex labeling.2

Figure 1 gives an example of $(a,1)$-edge-antimagic vertex labeling of Bt_n. With Theorem 1 in hand and by using Lemma 3, we obtain the following result.

Theorem 2. *A triangular book Bt_n has a super $(3n + 6,0)$-edge-antimagic total labeling and a super $(n + 6,2)$-edge-antimagic total labeling for $n \geq 1$.*

Proof.
We have proved that the vertex labeling α_1 is a $(3,1)$-edge antimagic vertex labeling. With respect to Lemma 2, by completing the edge labels $p+1, p+2, \ldots, p+q$, we are able to extend labeling α_1 to a super $(a_1,0)$-edge-antimagic total labeling and a super $(a_2,2)$-edge-antimagic total labeling, where, for $p = n+2$ and $q = 2n+1$, the value $a_1 = 3n + 6$ and the value $a_2 = n + 6$. 2

Theorem 3. *A triangular book Bt_n has a super $(2n + 6,1)$-edge-antimagic total labeling.*

Proof. Label the vertices of Bt_n with $\alpha_2(x_i) = \alpha_1(x_i)$ and $\alpha_2(y_j) = \alpha_1(y_j)$, for $i = 1,2$ and $1 \leq j \leq n$; and label the edges with the following way.

$$\alpha_2(x_1x_2) = \begin{cases} \frac{5n+7}{2}; & \text{if } n \text{ is odd} \\ \frac{3n+6}{2}; & \text{if } n \text{ is even} \end{cases}$$

For n odd, any j, and $i = 1,2$

$$\alpha_2(x_iy_j) = \frac{(5-i)n - j + (8-i) + ((-1)^j + 1)n}{2} + \frac{(-1)^j + 1}{4}$$

For n even, any j and $i = 1,2$

$$\alpha_2(x_iy_j) = \frac{(5-i)n - j + (8-i) + ((-1)^{j+(i-1)} + 1)n}{2} + \frac{(-1)^{j+(i-1)} + 1}{4}$$

The total labeling α_2 is a bijective function from $V(Bt_n) \cup E(Bt_n)$ onto the set $\{1,2,3,\ldots,3n+3\}$. The edge-weights of Bt_n, under the labeling α_2, constitute the following sets:

$$W_{\alpha_2}(x_1x_2) = \begin{cases} \frac{7n+13}{2}; & \text{if } n \text{ is odd} \\ \frac{5n+12}{2}; & \text{if } n \text{ is even} \end{cases}$$

For n odd, any j, and $i = 1,2$, the edge-weights of x_i
$W_{\alpha_2}(x_iy_j) = \frac{(3+i)n+(10+i)+j+((-1)^j+1)n}{2} + \frac{(-1)^j+1}{4}$;
For n even, any j, and $i = 1,2$,

$$W_{\alpha_2}(x_iy_j) = \frac{(3+i)n+(10+i)+j+((-1)^{j+(i-1)}+1)n}{2} + \frac{(-1)^{j+(i-1)}+1}{4};$$

It is not difficult to see that the union of the set $w_{\alpha 2}$ equals to $\{2n + 6, 2n+7, 2n+8, \ldots, 4n+6\}$ and contains an arithmetic sequence with the first term $2n + 6$ and common difference 1. Thus α_2 is a super $(2n + 6,1)$-edge-antimagic total labeling. This concludes the proof. 2

4. Diamond Ladder Dl_n

Diamond ladder graph denoted by Dl_n is a connected graph with a vertex set $V(Dl_n) = \{x_i, y_i, z_j; 1 \le i \le n, 1 \le j \le 2n\}$ and an edge set $E(Dl_n) = \{x_i x_{i+1}, y_i y_{i+1}; 1 \le i \le n-1\} \cup \{x_i y_i; 1 \le i \le n\} \cup \{z_j z_{j+1}; 2 \le j \le 2n-2 \text{ for } j \text{ even}\} \cup \{x_i z_{2i-1}, x_i z_{2i}, y_i z_{2i-1}, y_i z_{2i}; 1 \le i \le n\}$. Thus $|V(Dl_n)| = p = 4n$ and $|E(Dl_n)| = q = 8n-3$.
If diamond ladder graph has a super (a,d)-edge-antimagic total labeling then it follows from Lemma 1 that the upper bound of d is $d \le 2$ or $d \in \{0,1,2\}$. The following lemma describes an $(a,1)$-edge-antimagic vertex labeling for diamond ladder.

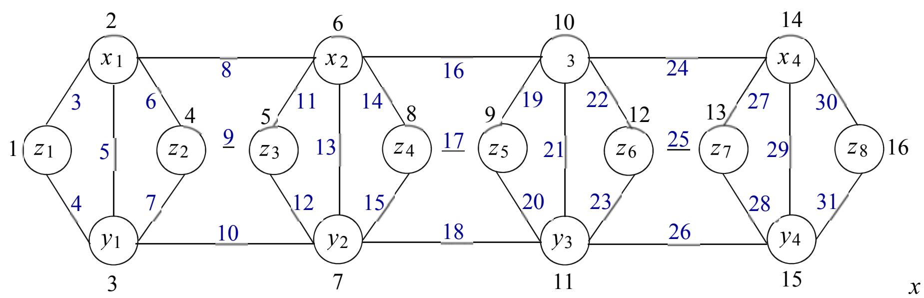

Figure 2. A (3,1)-edge antimagic vertex labeling of Dl_4

Theorem 4. *If $n \ge 2$ then the diamond ladder graph Dl_n has an $(a,1)$-edge-antimagic vertex labeling.*

Proof. Define the vertex labeling $\beta_1 : V(Dl_n) \to \{1,2,...,4n\}$ in the following way:
$\beta_1(x_i) = 4i-2$, for $1 \le i \le n$ $\beta_1(y_i) = 4i-1$, for $1 \le i \le n$

$$\beta_1(z_j) = 2j - \frac{((-1)^{j+1}+1)}{2}, \text{ for } 1 \le j \le 2n$$

The vertex labeling β_1 is a bijective function. The edge-weights of Dl_n, under the labeling β_1, constitute the following sets

$$\begin{aligned}
w_{\beta_1}(x_i x_{i+1}) &= 8i; \text{ for } 1 \le i \le n-1;\\
w_{\beta_1}(y_i y_{i+1}) &= 8i+2; \text{ for } 1 \le i \le n-1;\\
w_{\beta_1}(x_i y_i) &= 8i-3; \text{ for } 1 \le i \le n;\\
w_{\beta_1}(z_j z_{j+1}) &= 4j+1; \text{ for } 2 \le j \le 2n-2 \; j \text{ even};\\
w_{\beta_1}(x_i z_{2i-1}) &= 8i-5; \text{ for } 1 \le i \le n;\\
w_{\beta_1}(x_i z_{2i}) &= 8i-2; \text{ for } 1 \le i \le n;\\
w_{\beta_1}(y_i z_{2i-1}) &= 8i-4; \text{ for } 1 \le i \le n;\\
w_{\beta_1}(y_i z_{2i}) &= 8i-1; \text{ for } 1 \le i \le n;
\end{aligned}$$

It is not difficult to see that the union of $w_{\beta_1} = \{3,4,...,8n-1\}$ and consists of consecutive integers. Thus β_1 is a $(3,1)$-edge antimagic vertex labeling. 2 Figure 2 gives an example of a $(3,1)$-edge antimagic vertex labeling of Dl_4.
In similar way, with Theorem 4 in hand and by using Lemma 3, we obtain the following result.

Theorem 5. *If $n \geq 2$ then the graph Dl_n has a super* $(12n,0)$*-edge-antimagic total labeling and a super* $(4n+4,2)$*-edge-antimagic total labeling.*

Theorem 6. *If $n \geq 2$, then the graph Dl_n has a super* $(8n+2,1)$*-edge-antimagic total labeling.*

Proof. Label the vertices of Dl_n with $\beta_2(x_i) = \beta_1(x_i)$, $\beta_2(y_i) = \beta_1(y_i)$ and $\beta_2(z_j) = \beta_1(z_j)$, for $1 \leq i \leq n$ and $1 \leq j \leq 2n$; and label the edges with the following way.

$$
\begin{aligned}
\beta2(xixi{+}1) &= 12n - 4i - 1; \text{ for } 1 \leq i \leq n-1\\
\beta_2(y_iy_i{+}1) &= 12n - 4i - 2; \text{ for } 1 \leq i \leq n-1\\
\beta2(xiyi) &= 8n - 4i + 2; \text{ for } 1 \leq i \leq n\\
\beta_2(z_jz_j{+}1) &= 8n - 2j; \text{ for } 2 \leq j \leq 2n-2 \text{ even}\\
\beta2(xiz2i{-}1) &= 8n - 4i + 3; \text{ for } 1 \leq i \leq n\\
\beta2(xiz2i) &= 12n - 4i; \text{ for } 1 \leq i \leq n\\
\beta2(yiz2i{-}1) &= 12n - 4i + 1; \text{ for } 1 \leq i \leq n\\
\beta2(yiz2i) &= 8n - 4i + 1; \text{ for } 1 \leq i \leq n
\end{aligned}
$$

The total labeling β_2 is a bijective function from $V(Dl_n) \cup E(Dl_n)$ onto the set $\{1,2,3,\ldots,12n-3\}$. The edge-weights of Dl_n, under the labeling β_2, constitute the sets

$$
\begin{aligned}
W\beta_2(xixi{+}1) &= 12n + 4i - 1; \text{ for } 1 \leq i \leq n-1\\
W\beta_2(yiyi{+}1) &= 12n + 4i; \text{ for } 1 \leq i \leq n-1\\
W\beta2(xiyi) &= 8n + 4i - 1; \text{ for } 1 \leq i \leq n\\
W\beta_2(zjzj{+}1) &= 8n + 2j + 1; \text{ for } 2 \leq j \leq 2n-2, \text{ for } j \text{ even}\\
W\beta_2(xiz2i{-}1) &= 8n + 4i - 2; \text{ for } 1 \leq i \leq n\\
W\beta2(xiz2i) &= 12n + 4i - 2; \text{ for } 1 \leq i \leq n\\
W\beta_2(yiz2i{-}1) &= 12n + 4i - 3; \text{ for } 1 \leq i \leq n\\
W\beta2(yiz2i) &= 8n + 4i; \text{ for } 1 \leq i \leq n
\end{aligned}
$$

It is not difficult to see that the union of the the set $W_{\beta 2} = \{8n+2, 8n+3, \ldots, 16n-2\}$ contains an arithmetic sequence with the first term $8n+2$ and common difference 1. Thus β_2 is a super $(8n+2,1)$-edge-antimagic total labeling.
This concludes the proof. 2

Figure 3 gives an example of super (a,d)-edge antimagic total labeling of Dl_4 for $d = 1$.

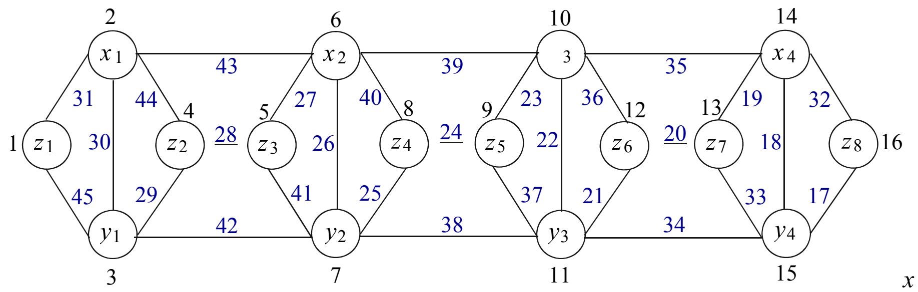

Figure 3. Super (34,1)-edge antimagic total labeling of Dl_4

5. Conclusion

In this paper, we have studied the existence of super antimagicness of two special families of graphs, namely triangular book and diamond ladder. The research shows the following results:

(1) The upper bound d of a super (a,d)-edge-antimagic total labeling at Bt_n and Dl_n is $d \leq 2$

(2) There are a super (a,d)-edge-antimagic total labeling of graph Bt_n and Dl_n, if $n \geq 1$ with $d \in \{0,1,2\}$.

Further interested research is then to answer following problem: If a graph Bt_n and Dl_n are super (a,d)-edge-antimagic total, are the disjoint union of multiple copies of the graphs Bt_n and Dl_n super (a,d)-edge-antimagic total as well? Therefore, we propose the following open problem.

Open Problem 1. *For the graph mBt_n, $n \geq 1$ and $m \geq 2$, determine if there exists a super (a,d)-edge-antimagic total labeling with any feasible upper bound d.*

Open Problem 2. *For the graph mDl_n, $n \geq 1$ and $m \geq 1$, determine if there exists a super (a,d)-edge-antimagic total labeling with any feasible upper bound d.*

References

[1] M. Baˇca, Dafik, M. Miller and J. Ryan, On super (a,d)-edge antimagic total labeling of caterpillars, *J. Combin. Math. Combin. Comput.*, **65** (2008), 61–70.

[2] M. Baˇca, Y. Lin, M. Miller and R. Simanjuntak, New constructions of magic and antimagic graph labelings, *Utilitas Math.* **60** (2001), 229–239.

[3] G.S. Bloom and S.W. Golomb, Applications of numbered undirected graphs, *Proc. IEEE*, **65** (1977) 562–570.

[4] R. Bodendiek and G. Walther, On (a,d)-antimagic parachutes, Ars Combin. 42 (1996), 129– 149.

[5] R. Bodendiek and G. Walther, (a,d)-antimagic parachutes II, Ars Combin. 46 (1997), 33–63.

[6] Dafik, Slamin, Fuad and Riris. 2009. *Super Edge-antimagic Total Labeling of Disjoint Union of Triangular Ladder and Lobster Graphs*.Yogyakarta: Proceeding of IndoMS International Conference of Mathematics and Applications (IICMA) 2009.

[7] Dafik, M. Miller, J. Ryan and M. Baˇca, Antimagic total labeling of disjoint union of complete *s*-partite graphs, *J. Combin. Math. Combin. Comput.*, **65** (2008), 41–49.

[8] Dafik, M. Miller, J. Ryan and M. Baˇca, On super (*a,d*)-edge antimagic total labeling of disconnected graphs, *Discrete Math.*, **309** (2009), 4909-4915.

[9] Dafik, M. Miller, J. Ryan and M. Baˇca, Super edge-antimagic total labelings of $mK_{n,n,n}$, *Ars Combinatoria* , **101** (2011), 97-107

[10] H. Enomoto, A.S. Llad´o, T. Nakamigawa and G. Ringel, Super edge-magic graphs, *SUT J. Math.* **34** (1998), 105–109.

[11] R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, The place of super edgemagic labelings among other classes of labelings, *Discrete Math.* **231** (2001), 153–168.

[12] R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, On super edge-magic graph, Ars Combin. 64 (2002), 81–95.

[13] N. Hartsfield and G. Ringel, Pearls in Graph Theory, Academic Press, Boston - San Diego New York - London, 1990.

[14] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.* **13** (1970), 451–461.

[15] G. Ringel and A.S. Llad´o, Another tree conjecture, *Bull. Inst. Combin. Appl.* **18** (1996), 83–85.

[16] R. Simanjuntak, F. Bertault and M. Miller, Two new (*a,d*)-antimagic graph labelings, *Proc. of Eleventh Australasian Workshop on Combinatorial Algorithms* (2000), 179–189.

[17] K.A. Sugeng, M. Miller and M. Baˇca, Super edge-antimagic total labelings, *Utilitas Math.*, **71** (2006) 131-141.

[18] W. D. Wallis, E. T. Baskoro, M. Miller and Slamin , Edge-magic total labelings, *Austral. J. Combin.* **22** (2000), 177–190.

STUDENT ENGAGEMENT MODEL OF MATHEMATICS DEPARTMENT'S STUDENTS OF UNIVERSITY OF INDONESIA

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Abstract. Every department in a University should strive to create students who feel engaged to their Department. An engaged student is a student who is satisfied, loyal, and attached to his/her Department. The model of the relationships among Student Engagement Components should be observed to obtain a better policy for evaluating and improving the department. Student Satisfaction can be used as a measuring instrument for a department's self-evaluation. High student Loyalty level will result in reliable resources for the department. While high student Attachment level will result in alumni who will still support the Department towards its improvement in the future. Hence, if a department possesses engaged students, the continuity of the department's progress can be achieved. Based on logical reasoning, a satisfied student usually is a loyal student, and a loyal student usually is an attached student. However this is not the case in reality. The model of the relationship is heavily dependent on students' characteristics, such as Batch, GPA, Parents' educations, Parents' occupations, and the Students' Moral Foundations. Students can be categorized into groups based on those characteristics. Each group of student characteristic will provide different relationship model of engagement components. This article will present a general conclusion of engagement component relationship model by taking into account the existence of student characteristic groups. The methods applied are Two Step Clustering and MetaSem. This article is written as a result of study conducted in Mathematics Department, University of Indonesia 2013.

*Key words and Phrases:*Student Engagement, MetaSem.

1. Introduction

A. Background

A College Department's success is very dependent on the quality of its students and alumni. High quality students will create a passionate learning processand creating valuable works. The same principle applies for the alumni. High quality alumni of a department will bring pride to the department. Especially if the alumni have emotional relation with the Department, they will be more than willing to support the developments of the Department. Hence the developments can be attained continuously.

In the banking sector, there is this term called *Customer Engagement* which refers to customers who are willing to defend and make every effort for the sake of certain company or product. There are three main components in Customer Engagement; Satisfied Customer, Loyal Customer, and Attached Customer (Customer with emotional bond with the company or product). Engaged customer is a very valuable asset to the company, especially as a free-of-charge marketing agent.

Borrowing the expression, the writer would like to introduce what is dubbed as Student Engagement. It refers to students who are willing to defend their Department in college both when they are still registered students and also when they have been alumni of the Department in the future. Student Engagement cannot be measured immediately because usually someone's engagement to Department is shown only after that person becomes an alumnus. Even so,an engaged student can be mold through the creation of essential components of Student Engagement. There are three components of Student Engagement, namely Satisfied Student, Loyal Student, and Attached Student (student with emotional bond with the Department where he/she is studying)

Satisfied student is student who feels that his/her expectation of the Department is in accordance with the reality. Students' satisfaction comprises of satisfaction towards learning materials, lecturers, teaching/educational system, facilities, and campus environment. Students' satisfaction towards the Department can be utilized as a measurement tool of Department's service quality.

Loyal student is student who is willing to participate and support activities initiated by the Department and is willing to do the best for in order to improve the Department. A loyal student will be an asset and a very much needed human resource to the Department.

Attached student is student who has emotional bond with his/her Department. Attachment is something, such as a tie, band, or fastener, which attaches one thing to another. A student with high attachment level to the Department will have a sense of belonging towards the Department. He/she will be proud when the Department is deemed well by other parties or is obtaining a certain accomplishment. He/she will also be saddened over the fact that the Department fails to win certain achievement, etc.

Due to the fact that Student Engagement can be developed through the shaping of its components, namely Satisfaction, Loyalty, and Attachment, thus the relationship model of the Student Engagement components ought to be evaluated in order to ensure the Department's ability to determine which component(s) should be first in line to be fixed in attempt to increase the Students' Engagement level.

Based on logical understanding, the three components of Student Engagement will form a certain relationship model. A satisfied student will usually be a loyal student, and a loyal student will usually be an attached student to his/her Department. If the relationship model is assumed as so, thus in order to obtain an Engaged student, the Department needs to put increasing student satisfaction level on the top list. If the students are satisfied, he will be loyal and followed by being attached to his/her Department. If the three components can be improved, there will be a higher probability that the Department get an Engaged Student.

However, in reality the relationship among the Student Engagement components are not always in that model. There are students who are satisfied but not loyal to the Department, and there are also ones who are loyal but not attached to the Department. If the relationship model is not as so, the Department have to find ways in order to obtain Engaged Student. The difference in this relationship model is very likely to be influenced by the students' characteristics, such as batch, GPA, parents' occupation, parents' educational background, and the moral foundation of the students. Based on those characteristics, the writer attempted to classify the students into several different categories in which students with the same characteristics are classified in the same categories, and those with different characteristics are classified into different categories. This relationship model of Student Engagement components needs to be examined for each of the student characteristic groups to finally attain a general model that will be used as a reference in generating regulations that will help increasing Student Engagement level towards the Department.

B. Research Purpose

To attain a general relationship model of the Student Engagement components by taking into account the existence of student characteristics groups.

C. Research Method

- **a.** Sample : Students of the Departement of Mathematics University of Indonesia, batch 2010 (28 students), 2011 (40 students), 2012 (45 students) which are sampled by purposive sampling method. Data collection technique used is questionnaire distribution.
- **b.** Measurement instrument used is measurement tool with Likert scale to measure satisfaction, loyalty, attachment and moral foundation of the students. This particularmeasurement tool is designed based on the theoretical elements used as its foundation.
  The reliability of the measurement tool is 0.759 and every single item is valid item.
- **c.** Data analysis methods used are Clustering and MetaSem (combination of Meta Analysis and Structural Equation Model)

2. Main Results

A. Descriptive Statistics

- Batch: 2010 (28 students), 2011 (40 students), 2012 (45 students)
- GPA: ≤ 2.75 (8 students), >2.75 (104 students), missing (1 students)
- Parent's education: ≤ high school (42 students), > high school (67 students), missing (4 students)
- Parent's job: Government Employee (26 students), Private Employee (32 students), Entrepreneur (31 students), others (19 students), missing (5 students)
- Average Satisfaction score/item : 3.08
- Average Loyalty score/item : 3.54
- Average Attachment score/item : 3.82
- Average Moral Foundation score/item : 3.32

Since a respondent's answer is considered high when the score is > 3.5, it can be concluded that the respondents' have an average of low Satisfaction, high Loyalty, high Attachment, and low Moral Foundation.

B. Student Characteristic Groups Profile based on Batch, GPA, Occupation, Eduction, and Moral Foundation

By using Two Step Clustering method, the writer attempted to classify the respondents based on Batch, GPA, Parent's education, Parent's job and Moral Foundation variables but unfortunately there was only one variable that was able to separate the respondents into groups with the same characteristics within the same categories and different characteristics across categories. The variable that was able to create the aforementioned student characteristic groups is the Batch variable. Subsequently to find the relationship model of Student Engagement components, Batch variable will be taken into account for the analysis.

C. Relationship Model of Student Engagement Components

The relationship of Student Engagement components by taking batch into account will be attained by using Meta SEM method. This method is a combination of Meta Analysis method and Structural Equation Model method.

Structural Equation Model

Student Engagement components (Satisfaction, Loyalty, Attachment) are latent variables measured by indicators in the form of items containing all of the theoretical concepts. The items have been verified to be having Alpha Chronbach Reliability = 0.759 and every single one is a valid item.

Structural Equation Model method is set up by modeling the concept in the form of path diagram. Parameters are estimated by matching the correlation model matrix with the correlation matrix obtained from the samples of indicator-variables forming latent variables. In the working process, statistics software, will automatically transform raw data to covariance/correlation matrix form. Consequently in Structural Equation Model analysis, data can be input in the form of raw data, covariance matrix or correlation matrix of the indicator-variables. In this paper, correlation matrix will be used as input data.

The correlations among indicator-variables in each batch are as follows:

Batch 2010:

$$\begin{bmatrix} & Q7 & Q22 & Q25 & Q2 & Q11 & Q12 & Q15 \\ Q7 & 1 & & & & & & \\ Q22 & 0.089 & 1 & & & & & \\ Q25 & 0.315 & 0.379 & 1 & & & & \\ Q2 & 0.302 & 0.246 & 0.06 & 1 & & & \\ Q11 & 0.307 & 0.415 & 0.268 & 0.036 & 1 & & \\ Q12 & 0.317 & 0.215 & 0.442 & 0.087 & 0.369 & 1 & \\ Q15 & 0.477 & 0.197 & 0.508 & 0.438 & 0.29 & 0.472 & 1 \end{bmatrix}$$

Batch 2011:

	$Q7$	$Q22$	$Q25$	$Q2$	$Q11$	$Q12$	$Q15$
$Q7$	1						
$Q22$	0.276	1					
$Q25$	0.386	0.402	1				
$Q2$	0.527	0.072	0.011	1			
$Q11$	0.002	0.017	0.005	0.123	1		
$Q12$	0.211	0.016	0.492	0.304	0.271	1	
$Q15$	0.368	0.140	0.445	0.521	0.300	0.549	1

Batch 2012:

	$Q7$	$Q22$	$Q25$	$Q2$	$Q11$	$Q12$	$Q15$
$Q7$	1						
$Q22$	0.579	1					
$Q25$	0.378	0.414	1				
$Q2$	0.095	0.119	0.153	1			
$Q11$	0.009	0.181	0.110	0.163	1		
$Q12$	0.419	0.370	0.511	0.154	0.095	1	
$Q15$	0.100	0.141	0.234	0.061	0.047	0.301	1

Subsequently, to find the relationship of the Student Engagement components, the overall correlation that represents the existing batch correlation should first be obtained. Overall correlation is obtained using MetaAnalysis method. The coefficient of the overall correlation will be used as an entry for input matrix in Structural Equation Model.

Meta Analysis Method

As previously mentioned, the respondents of this research are batch 2010, 2011, and 2012 students of the Mathematics Department, University of Indonesia. By taking batch into account, overall correlation of the indicator-variables will be obtained based on the correlation among the indicator- variables of each batch. Overall correlation will be calculated using Meta Analysis method.

Assumed ρ_i is correlation coefficient between two indicator variables in batch-i.

Define $Z_i = 0.5 \ln \frac{1+\rho_i}{1-\rho_i}$

it can be shown thatZ_i has variance $\approx N_i - 3$.

Before finding overall correlation, the equality of ρ, is evaluated by testing hypothesis $H_0 : \rho_1 = \rho_2 = \rho_3$.

The conclusion is "H_0 is not rejected".

Define $\widehat{Z_\iota} = 0.5 \ln \frac{1+r_i}{1-r_i}$; r_i is correlation estimation of ρ_i.

Compute $Z^* = \frac{\sum w_i \widehat{Z_\iota}}{\sum w_i}$ where $w_i = \frac{1}{N_i - 3}$

$$r^* = \frac{e^{2z} - 1}{e^{2z} + 1}$$

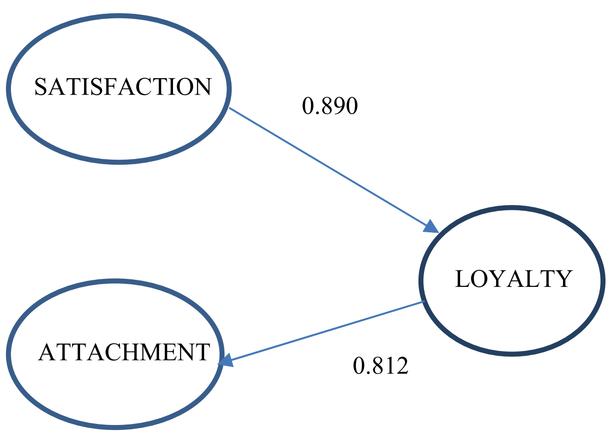
r^* is the overall correlation we want to obtain.
Values of r^* for every pair of indicator-variabeles will be used as entry matrix of input matrix in Structural equation Model.
By using Meta Analysis, the following result of overall correlations of each pair of the indicator-variables is obtained

	Q7	Q22	Q25	Q2	Q11	Q12	Q15
Q7	1						
Q22	0.2884	1					
Q25	0.3705	0.3244	1				
Q2	0.1416	0.3143	0.3540	1			
Q11	0.3952	0.1620	0.2426	0.2000	1		
Q12	0.1656	0.070	0.1503	0.2000	0.1657	1	
Q15	0.1096	0.1708	0.4231	0.2712	0.2316	0.4548	1

The above mentioned overall correlation matrix is then used as input matrix in Structural Equation Model.

Meta SEM Method

Meta SEM method is a combination of Meta Analysis method and SEM method. Meta Analysis method is used to find the overall correlation and SEM is used to find the relationship model of the Student Engagement components by utilizing overall correlation matrix as input matrix.
The best model of Student Engagement Model obtained is as follow:



Student Engagement Model

The numbers in the above diagram presents standardized effect, thus the values can be compared. Goodness of fit of Structural Equation Model can be examined using several different methods. In this paper, Goodness of fit of the obtained Model will be examined using

- Chi Square test, model is fit if p-value > 0.05
- CFI : model is fit if CFI > 0.9

- CMIN/DF : model is deemed fit if CMIN/DF lies between 1 – 3
- RMSEA : model is fit if RMSEA = 0.05 – 0.08

From the result of data processing, Goodness of fit of the model is obtained:
P - Chi Square = 0.056 > 0.05
CFI = 0.941 (standard > 0.9)
C MIN/DF = 1.718 (standard 1 – 3)
RMSEA = 0.08 (standard 0.05 – 0.08)
Conclusion: Model is fit

Note: When using Structural Equation Model method, the use of overall correlation matrix as input matrix did not allow the writer to examine multivariate normality assumption.

3. Concluding Remarks

Based on the data analysis above, it can be concluded that the assumption about how satisfied student will result in loyal student, loyal student will result in attached student is true for student of the Mathematics Department, University of Indonesia. While satisfied students does not automatically become attached student. Therefore the Department must pay a good attention to both satisfaction and loyalty to achieve attached students.

The numbers in the above mentioned diagram presents standardized effect, thus the values can be compared. It can be seen that the influence of Satisfaction towards Loyalty is higher than the influence of Loyalty towards Attachment. Therefore it is essential for the Department to prioritize its effort in improving students' satisfaction. For that reason, the Department is required to understand the needs of the students and look into the currently available services. A separate survey is needed for to learn about the students' needs and to compare it with the currently available services. The purpose is so that the Department can evaluate what segments of the Department to be improved and what segments to be preserved. As a result, students' satisfaction level can be optimized.

Aside from that, we can see from the model that Loyalty is another influencing factor to Attachment, while Satisfaction does not directly influence Attachment. Thus, to get students who are attached to the Department, an effort to enhance students' loyalty in order that the students are willing to do their best for the sake of the Department is needed. By the existence of loyal students, attached students will exist accordingly.

The existence of satisfaction, loyalty and attachment in the students create engaged students. Engaged students are extremely valuable assets for the Department in order to maintain the incessant improvement of the Department's quality.

Acknowledge:
Thank you for my students who have helped me to collect the data.

References

[1] Cheung, M. L. W., Chan, W. (2005).*Meta-Analytic Structural Equation Modeling: A Two-Batch Approach.* Psychological Methods2005, Vol. 10, \No. 1 : 40 – 64

[2] Bollen Kenneth A. (1989)*Structural Equation with Latent Variables;* John Wiley & Sons

[3] Rencher AC.(2002).*Methods of Multivariate Analysis*, John Wiley & Sons

[4] Virginia Gunawan.(2013). *Hubungan Pola Asuh Orang tua dan Fondasi Moral pada Remaja* (thesis)

[5] Zhilin Young, Robin T Pererson. (2004).*Customer Perceieved Value, Satisfaction and Loyalty: the Role of Switching cost Psychology & Marketing, Vol. 21(10): 799-822 (October 2004), Wiley InterScience*(www.interscience.com)

DESIGNING ADDITION OPERATION LEARNING IN THE MATHEMATICS OF GASING FOR RURAL AREA STUDENT IN INDONESIA

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Abstract. Learning number operations at the primary school is important for learning other subjects. It's because learning number operations tends to an understanding of notation, symbols, and other forms to represent (reference number), so it can support the students' thinking and understanding, to solve their problems [6]. Several studies on mathematics learning for Papua's student indicate that students have difficulty in understanding the concept of number operations [13]. It's supported by the results of rural area's student, namely Ambon, Serui, and Sorong Selatan, classroom observations toward to learning number operations conducted by researchers in pre-test. Students are more likely to be introduced by the use of the formula without involving the concept itself and learning number operations separate the concrete situation of learning [8,9]. Addition operation is number operation which first must be mastered before student learns another number operations starting from introduction of number [10]. This underlies the researcher to try designing addition operation learning in the mathematics of GASING (Math GASING) for them, which always starts from the concrete (informal level) to the abstract (formal level) [11]. Concrete means real, can be touched, seen, and explored. Once students are able to relate the concrete forms to abstract (mathematics symbols), they are required to do much exercise (drill) with mental arithmetic namely *mencongak* [8]. The purpose of this study is to look at the role of learning addition operation in Math GASING in helping students' understanding and mastering of the addition concept from the informal (concrete) into formal level. The research method used is a design research with preliminary design, teaching experiments, and retrospective analysis stages [1,3,5]. This study describes how the Math GASING make a real contribution of students understanding in the concept of addition operation. The whole strategy and model that students discover, describe, and discuss the construction or contribution shows how students can use to help their initial understanding of the addition concept. The stages in the learning trajectory has important role in understanding the addition concept from informal to formal level [2,4].

Key words and Phrases: Design Research, Math GASING, Addition Operation, Rural Area's Student.

1. Introduction

Professional teacher as the product of reform in education must have higher education and be able to innovate in teaching and learning [7]. So, every prospective teacher should be able to prepare themselves to become professional teachers to equip themselves with a high education and knowledge of the learning and teaching process. In the other hands, prospective teachers who come from rural areas have a few access to get decent education and information as requirements to become a professional teacher. Surya College of Education has responsibility for it. Here, they get a great education to become a professional teacher including mathematics teacher. In addition, Surya [13] have made and apply a learning innovation in mathematics education, named Math GASING. This learning has been applied to student from Papua, which began with the introduction of number and number operations, and produce many Olympic champions both nationally and internationally.

Furthermore, learning number operations at the primary school is important for learning other subjects. It's because learning number operations tends to an understanding of notation, symbols, and other forms to represent (reference number), so it can support the students' thinking and understanding, to solve their problems [6]. Based on the results of several previous studies show students have difficulty to understand the number operation concept [8,9]. It's supported by the results of rural area's student, namely Ambon, Serui, and Sorong Selatan, classroom observations toward to learning number operations conducted by researchers in pre-test. Students are more likely to be introduced by the use of the formula without involving the concept itself and learning number operations separate the concrete situation of learning. Addition operation is number operation which first must be mastered before student learns another number operations starting from introduction of number [10]. This underlies the researcher to try designing addition operation learning in the mathematics of GASING (Math GASING) which always starts from the concrete (informal level) to the abstract (formal level) for matriculation prospective teachers students at Surya College of Education Tangerang derived from Ambon, Serui, Yapen, and South Sorong, Papua.

2. Theoretical Framework

In this study, the literature on Math GASING and addition operation was learn to see the typical learning processes used by real situations (concrete) to abstract with the steps that has been in the design.

Math GASING

Surya and Moss [13] stated that GASING has several basic premises. First is that there is no such thing as a child that cannot learn mathematics, only children that have not had the opportunity to learn mathematics in a fun and meaningful way. Second is that mathematics is based on patterns and these patterns make math understandable. Third is that a visual context to mathematical concepts should come before the symbolic notation. Lastly is that mathematics is not memorization, but knowing basic facts comes easily with a conceptual and visual understanding. Memorization of basic mathematics facts is easy if it is based on conceptual learning and visual representations. Additionally, Shanty and Wijaya [11] describes that in Math GASING, the learning process make students learning easy, fun, and enjoyable. Easy means the students are introduced to mathematical logic that is easy to learn and to remember. Exciting means the students have motivation

which comes from by them to learn mathematics (intrinsic factor). Fun is more in the direction of outside influences such as visual aids and games (extrinsic factor). In the other hand, Prahmana [8] had been conducted research for division topic in Math GASING, where the learning process begins with the activities share sweets fairly, then move into the process of how each student gets distributed sweets after a fair amount of candy (concrete), ranging from division without remainder to division with remainder, and ends with the completion of division operation in Math GASING (abstract). Math GASING shows
how to change a concrete sample into an abstract symbol so the students will be able to read a mathematical pattern, thus gain the conclusion by themselves

Addition Operations in Math GASING

Math GASING as one of innovations in learning mathematics offers critical point in its learning process. When studying a topic in Math GASING, there is a critical point that we must pass that is called GASING's critical point. After reaching this critical point, students will not be difficult anymore to work on the problems in that topic [13]. The critical point in learning addition is addition of two numbers between 1 and 10 with a sum less than 20. In the other words, when a student has mastered addition of two numbers between 1 and 10 with a sum less than 20, the student can learn a variety of addition operation problems more easy.

The Hypothetical Learning Trajectory (HLT) in this study had several learning goals expected to be reached by the students. To reach the goals formulated, researcher designs a sequence of instructional learning for learning addition in Math GASING on the following diagram.

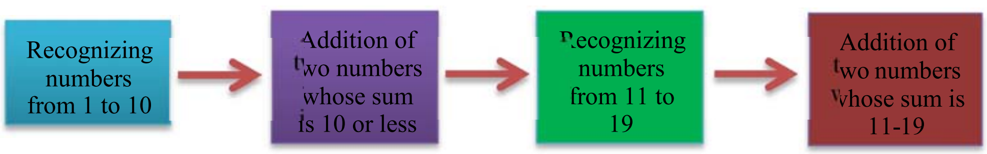

Figure 2.1 the HLT of Learning Addition in Math GASING

The explanation of Figure 2.1 is as follows:

1. Students are recognizing numbers from 1 to 10 by using their fingers. For example, teacher introduced the notation of number "1" by showing 1 index finger or 1 thumb or 1 middle finger or 1 ring finger or 1 little finger; the notation of number "2" by showing 1 index finger and 1 thumb or 2 index fingers or 2 thumbs, and various other variations; the notation of number "3", "4", until "10" by using various other variations from their finger.
2. Students learn the addition of two numbers whose sum is 10 or less process by using their fingers. For example, teacher showed 3 fingers on the right hand and said "these are three fingers", and then showed 2 fingers on the left hand and said "these are two fingers". After that, she combined the fingers on both hands together and said "these are five fingers", so "three plus two is equal to five". Lastly, students learned how to write in abstract symbols: $3 + 2 = 5$. Teacher showed all various combination of addition from 2 to 10.
3. Students are recognizing numbers from 11 to 19 by using "number card". For example, "black card" as tens and "white card" as unit. The teacher showed 1 black card and said "this is ten", and then showed 1 white card and said "this is one". After that, she combined the two cards that consists of 1 black card and white card, and said "this is eleven". This way is able to make the students

imagine that eleven consists of ten and one. Teacher showed all numbers from 1119 by using black card and white card.

4. Students learn the addition of two numbers whose sum is 11-19 by using "number card". The learning process consists of some addition types namely 10+, 9+, 8+, 7+, and 6+ and starts from 10+ type where student add the number 10 with numbers from 1 to 9, to 6+ type (see table 2.1). They also learn about the commutative of addition according to table 2.1.

Table 2.1 Addition of two numbers whose sum is between 11 and 19

10 +	**9 +**	**8 +**	**7 +**	**6 +**
10 + 1 =				
10 + 2 =	9 + 2 =			
10 + 3 =	9 + 3 =	8 + 3 =		
10 + 4 =	9 + 4 =	8 + 4 =	7 + 4 =	
10 + 5 =	9 + 5 =	8 + 5 =	7 + 5 =	6 + 5 =
10 + 6 =	9 + 6 =	8 + 6 =	7 + 6 =	6 + 6 =
10 + 7 =	9 + 7 =	8 + 7 =	7 + 7 =	
10 + 8 =	9 + 8 =	8 + 8 =		
10 + 9 =	9 + 9 =			

Research Question

Based on a few things mentioned in the introduction above, then researcher formulates a research question in this study, as follows:

"How is student learning trajectory of learning addition in Math GASING, which evolved from informal to formal level for rural area's student at Surya College of Education?

3. Methods

This study uses a design research approach, which is an appropriate way to answer the research questions and achieve the research objectives that start from preliminary design, teaching experiments, and retrospective analysis [9]. Design research is methodology that has five characteristic, which is interventionist nature, process oriented, reflective component, cyclic character, and theory oriented [1]. To implementation, design research is a cyclical process of thought experiment and instruction experiments [3]. There are two important aspect related to design research. There are the Hypothetical Learning Trajectory (HLT) and Local Instruction Theory (LIT). Both will be on learning activities as learning paths that may be taken by students in their learning activities.

According to Freudenthal in Gravemeijer & Eerde [5], students are given the opportunity to build and develop their ideas and thoughts when constructing the mathematics. Teachers can select appropriate learning activities as a basis to stimulate students to think and act when constructing the mathematics. Gravemeijer [4] states that the HLT consists of three components, namely (1) the purpose of mathematics teaching for students, (2) learning activity and devices or media are used in the learning process, and (3) a conjecture of understanding the process of learning how to learn and strategies students that arise and thrive when learning activities done in class.

For the data, researchers have collected research data is derived from multiple sources of data, to get a visualization of the students' mastery of basic concepts of addition operations, namely video recording, documentation (learning activities photo), and the data is written (the results of students' answers and observation sheet). Furthermore, the data were analyzed retrospectively with HLT as a guide. In addition, these studies have been completed in 2 days in the first semester of academic year 2013/2014 with the subjects are 11 matriculation prospective teachers students at Surya College of Education Tangerang derived from Ambon, Serui, Yapen, and South Sorong, Papua, and also a teaching assistant who acted as a model teacher.

This study consists of three steps done repeatedly until the discovery of a new theory that a revision of the theory of learning is tested. Overall, the stages that will be passed on this research can conclude in the form of the following diagram in Figure 3.1. [9]:

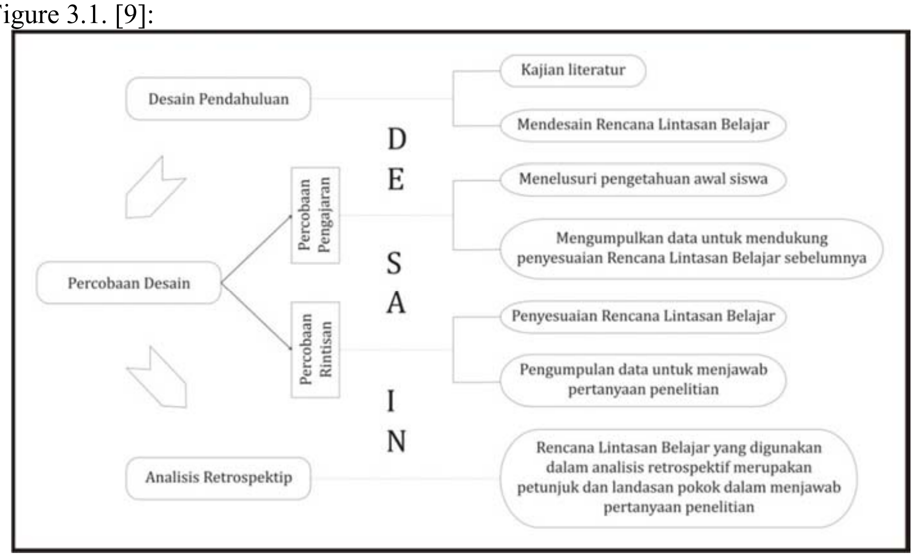

Figure 3.1 Phase of the design research

4. Results and Analysis

The learning activities start from recognizes number between 1 and 10 using students'
fingers to introduce the concept of number in concrete level as sum of their fingers. Furthermore, Students learn the addition of two numbers whose sum is 10 or less process by using their fingers. Lastly, students are recognizing numbers from 11 to 19 by using "number card" and learning the addition of two numbers whose sum is 11-19 by using "number card" that consist of black card as tens and white card as units. At the end of the second meeting, students do mental arithmetic activity namely *mencongak* as one of assessment process in this learning activities and exercise by using student evaluation sheet. As a result, students was able to master the addition operation in Math GASING seen from the results of the final evaluation and was pleased to learn Math GASING can be seen from the comments of students who wish to abandon the old way of learning mathematics. The results of this study indicate that learning design of addition operation in Math GASING have a very important role as the starting point and improve students'

motivation in learning addition operation.
For more details, researchers will discuss the results of this study, which is divided into three stages that are called preliminary design, teaching experiments, and retrospective analysis.

Preliminary Design

At this stage, researcher is beginning to implement the idea of addition operation in Math GASING by reviewing the literature, conducting observations in matriculation class, and ends with designing hypothetical learning trajectory (HLT), as shown in Figure 2.1. A set of activities for learning addition operation in Math GASING has been designed based learning trajectory and thinking process of students who hypothesized. The instruction set of activities has been divided into four activities that have been completed in 2 meetings, start from recognize numbers between 1 and 10 using students' fingers as the concrete form, learn the addition of two numbers whose sum is 10 or less process by using their fingers, recognize numbers from 11 to 19 by using "number card", learn the addition of two numbers whose sum is 11-19 by using "number card" that consist of black card as tens and white card as units, do a variety of fun activities that make students happy in the learning process, and end with the evaluation process.

Teaching Experiment

In teaching experiment, researcher tests the learning activities have been designed in the preliminary design stage. When the teacher models have started to see students do not get excited, then the teacher models provide educational games that make fun learning activities, because it is becoming one of characteristics in Math GASING learning process. There are four activities in this stage. First, teacher introduced the notation of number "1" until "10" by showing her finger and student recognize it by using their fingers as the concrete form using various other variations from their fingers (see in Figure 4.1).

Figure 4.1 students recognize numbers between 1 and 10 using their fingers

Second, students learn the addition of two numbers whose sum is 10 or less process by using their fingers. Student showed 4 fingers on the right hand and said "this is four", and then showed 2 fingers on the left hand and said "this is two". After that, she combined the fingers on both hands together and said "this is six", so "four plus two is equal to six". Lastly, students write in abstract symbols: 4 + 2 = 6. Teacher showed all various combinations of addition from 2 to 10 (see in Figure 4.2).

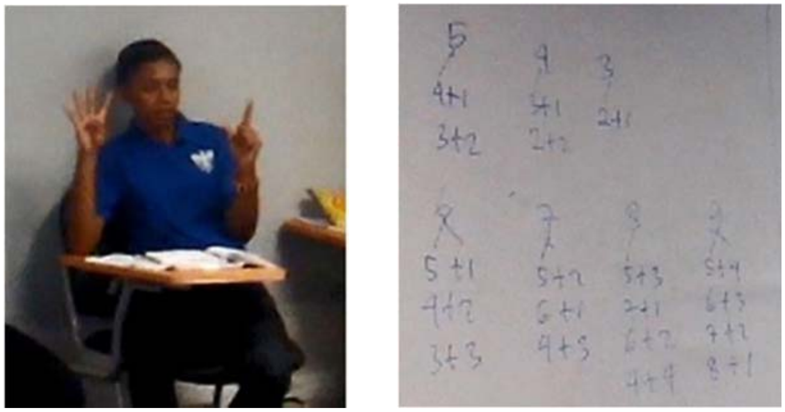

Figure 4.2 students learn addition of two numbers (left) and all various combinations (right)

Third, students are recognizing numbers from 11 to 19 by using “number card” that consists of “black card” as tens and “white card” as unit. Student showed 1 black card and said, “This is ten”, and then showed 1 white card and said, “This is one”. After that, she combined the two cards of 1 black card and 1 white card, and said, “This is eleven”. Teacher showed all numbers from 11-19 by using black card and white card (see in Figure 4.3).

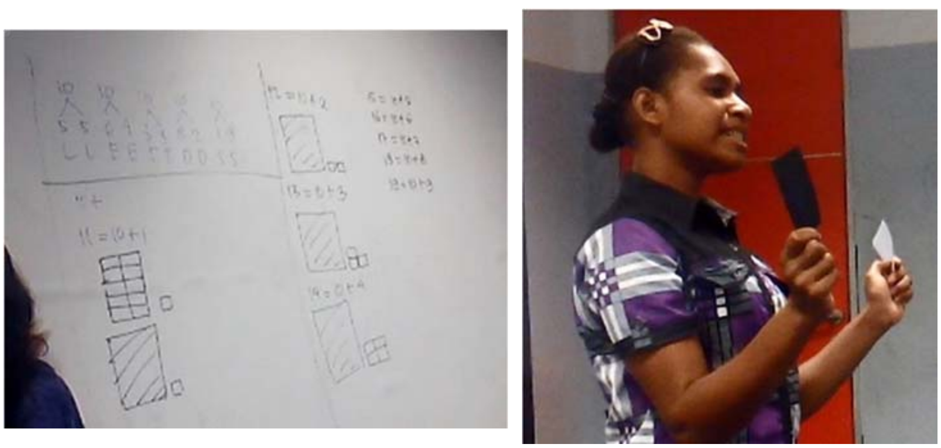

Figure 4.3 the concrete form of 10+ type (left) and students presented the number card (right)

Lastly, students learn the addition of two numbers whose sum is 11-19 by using “number card”. The learning process consists of some addition types namely 10+, 9+, 8+, 7+, and 6+ and starts from 10+ type where student add the number 10 with numbers from 1 to 9, to 6+ type (see table 2.1). They also learn about the commutative of addition according to table 2.1. Student count nine white card plus three white card, and then count all white card that she gets. After that, she switch 10 white card with 1 black card and now she gets 1 black card and 2 white card, and said, “This is twelve” (see in Figure 4.4).

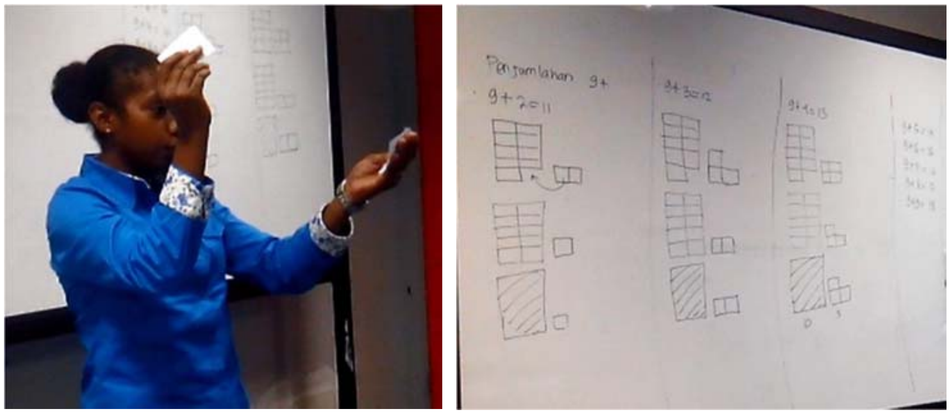

Figure 4.4 students learn addition of two numbers (left) and all various combinations 9+ (right)

The learning process in this study ends with 2 forms of evaluation. First, students are given about spontaneously problem in front of class and direct answer that problem on whiteboard. Second, students are given a worksheet that consists of many questions about addition and should be able to finish it within a few minutes (see in Figure 4.5).

The results of the evaluation process are quite amazing that all students get satisfactory results and are able to explain it either concretely or abstractly.

Figure 4.5 the first evaluation form (left) and the second evaluation form (right)

Retrospective Analysis

Addition process in Math GASING is different with addition process in mathematics in general. As a result, all activities which have been designed can be used to answer the research question above. The activities are as follows:

1. Learning trajectory which has been modeled in Figure 2.1 are the activities undertaken in this study to guide students mastered addition operation. So that, researcher designed an activity using students' fingers to recognize numbers from 1 to 10 and to learn by using their fingers too. The goal is that students are able to imagine the concrete form of number notation between 1 and 10 and the addition of two numbers whose sum is 10 or less process. Fingers are the greatest learning tool to introduce the concrete form of number between 1 and 10 and that addition. Next, to introduce a number greater than

10 and its sum, researcher uses a number card starting from units, tens, hundreds, etc. by using different colors such as white card for the unit, the black card to tens, the orange card to hundreds, etc.

2. Furthermore, from these activities, teachers guide students toward the concept of number notation and addition using their fingers and number card. As a result, when students have mastered the addition process of two numbers whose sum is between 11 and 19, they were able to complete the various forms of addition operations more easily, using the addition process in Math GASING. So that, they can do *mencongak* to solve all addition problems given. Addition process in Math GASING is starting from "front addition" where addition process start from left to right and "scratch system" for the addition of two numbers whose sum is between 10 and 19. For more details, see in Figure 4.6 below.

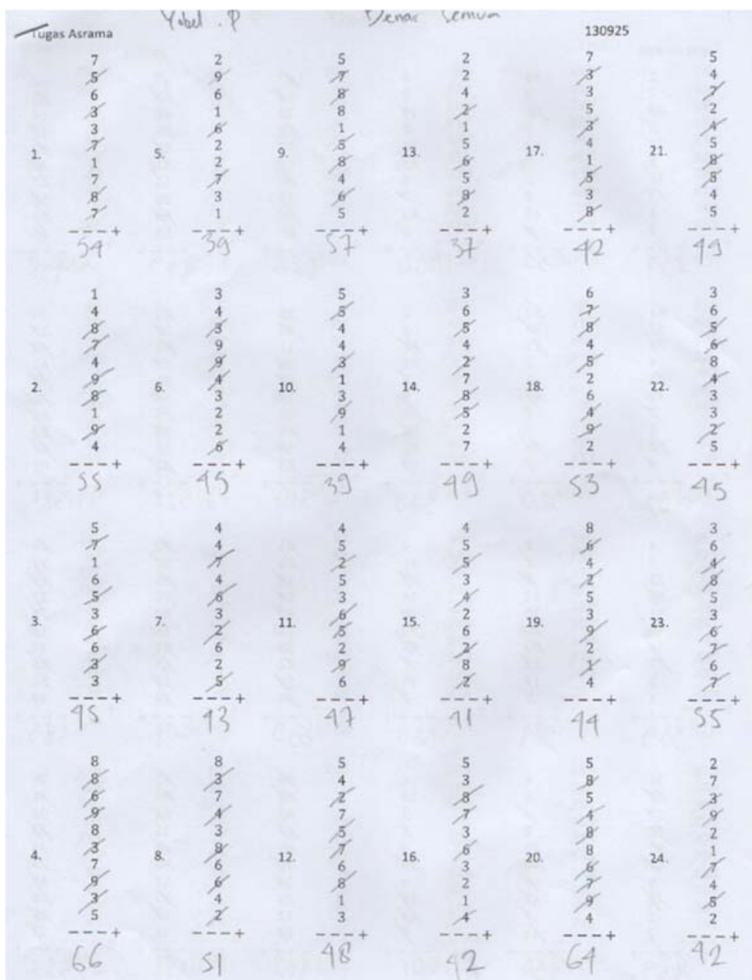

Figure 4.6 Yobel's answer sheet using "scratch system"

3. In Figure 4.6, Yobel's answer sheet shows addition process in Math GASING where every addition of two numbers whose sum is more than or equal to 10

then he did ”scratch” on that number and so on until finished and then count the number of scratch made and write it followed by the last number that he count. For example, based on problem 4, 8 + 8 is equal to 16. It means he must do scratch on “8”, and then 6 + 6 is equal to 12. It means he must do scratch again on “6”, and then 2 + 9 is equal to 11. It means he must do scratch on “9”, and then 1 + 8 is equal to 9. It means he don’t do scratch on “8”, and continue until finished. Lastly, he get 6 scratch and “6” as the last number that he count. So, he can write “66” as the result of that problem. In the other hands, Figure 4.7 shows that rosita can solve 100 addition problems only 7 minutes and gets 1 mistake using “front addition”. It is apparent that addition process in Math GASING is much more effective than the usual process of addition, when students have mastered the addition of two numbers whose sum is between 11 and 19.

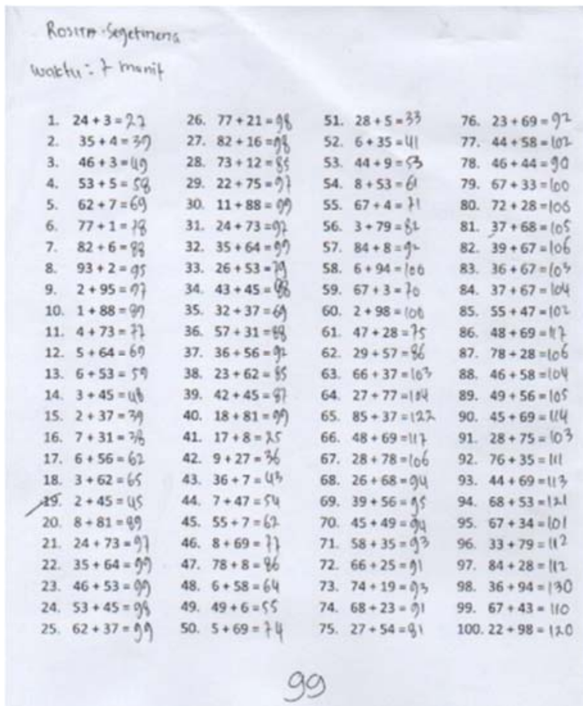

Figure 4.7 Rosita’s answer sheet using “front addition”

4. Based on all the activities above, it can be seen that the students have gone through the process of activity based on experience using their fingers and “number card”, moving toward a more formal, the understanding of formal level from the critical point, and then reached into the formal level desired as the ultimate goal of this learning activities.
5. In the design of this study, researcher used the learning steps of addition operation in Math GASING as shown in Figure 2.1. When the activity takes place, the dialogue is very good in the process of introducing the basic concepts of addition operations. In the dialogue, it seems that students feel learning addition operation in Math GASING looks so easy and so much fun. As a result, the learning process can guide students in understanding addition operations. It can also be seen from the student evaluation of learning addition process given by the teacher to evaluate student understanding. As a result, students seemed to be able to apply addition operation process in solving each

problem is given in terms of evaluation (see in Figure 4.8). Therefore, it can be seen that learning addition operation in Math GASING can use to raise students' understanding in integer addition operations or in other words, the design of this study can be used as the starting point of learning addition operations.

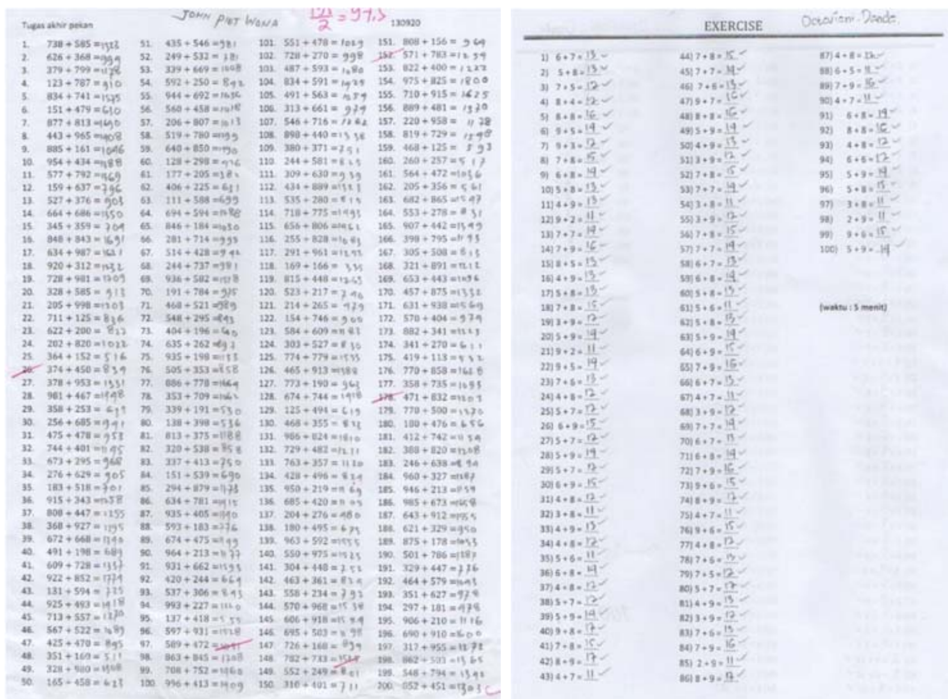

Figure 4.8 some student's answer sheets

5. Concluding Remarks

Based on the result and analysis of this research that has been described above, researcher can conclude that the learning of addition operation in Math GASING have a very important role as the starting point and improve students' motivation in learning addition operation. In addition, the activities that have been designed in such way those students find the concept of addition operation starting from recognizing number between 1 and 20 to calculating addition operation of two numbers whose sum is between 11 and 19 which is the critical point of addition operation in Math GASING. This process begins with the activities recognize number between 1 and 19 using their finger and number card (black and white card), and then move into the process of how to calculate two numbers addition operation of two numbers whose sum is between 11 and 19. Lastly, each student can do *mencongak* for any given addition problem and resolve many addition questions very quickly and precisely where is both of this are one of assessment forms in Math GASING.

Acknowledgement. Researchers would like to thank Petra Suwasti as a research assistant for their contribution in order to collect data and to be a model teacher in

this research.

References

[1] Akker, J.V.D., Gravemeijer, K., McKenney, S., and Nieveen, N. (2006). *Education Design Research*. London: Routledge Taylor and Francis Group.

[2] Carpenter, T. P., & Fennema, E. (1992). Cognitively guided instruction: Building on the knowledge of students and teachers. In W. *Secada (Ed.), Curriculum reform: The case of mathematics in the United States. Special issue of the International Journal of Educational Research (pp. 457–470)*. Elmswood, NY: Pergamon Press, Inc.

[3] Gravemeijer, K. (1994). *Developing Realistic Mathematics Education*. Utrecht: Technipress, Culemborg.

[4] Gravemeijer, K. (2004). Local Instructional Theories as Means of Support for Teacher in Reform Mathematics Education. *Mathematical Thinking and Learning, 6(2), 105-128*, Lawrence Erlbaum Association, Inc.

[5] Gravemeijer, K., Eerde, D.V. (2009). Design Research as a Means for Building a Knowledge Base for Teaching in Mathematics Education. *The Elementary School Journal Volume 109 Number 5*.

[6] National Council of Teachers of Mathematics (NCTM). (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

[7] Ornstein, A.C., Levine, D.U., Gutek, G.L. (2011). *Foundations of Education*. (11th ed.). Belmont: Wadsworth.

[8] Prahmana, R.C.I. (2013). Designing Division Operation Learning in The Mathematics of Gasing. *Proceeding in The First South East Asia Design/Development Research (SEA-DR) Conference 2013, pp. 391-398*. Palembang: Sriwijaya University.

[9] Prahmana, R.C.I., Zulkardi, Hartono, Y. (2012). Learning Multiplication Using Indonesian Traditional Game in Third Grade. *Journal on Mathematics Education (IndoMS-JME) Vol. 3 No. 2*, pp. 115-132. Palembang: IndoMs.

[10] Reys, R. E., Suydam, M. N., Lindquist, M. M., & Smith, N. L. (1984). *Helping Children Learn Mathematics*. (5th ed.). Boston: Allyn and Bacon.

[11] Shanty, N.O., Wijaya, S. (2012). Rectangular Array Model Supporting Students' Spatial Structuring in Learning Multiplication. Journal on Mathematics Education (IndoMS-JME) Vol. 3 No. 2, pp. 175-186. Palembang: IndoMs

[12] Surya, Y. (2011). *Petunjuk Guru: Dasar-Dasar Pintar Berhitung GASING*. Tangerang: PT. Kandel.

[13] Surya, Y., Moss, M. (2012). Mathematics Education in Rural Indonesia. *Proceeding in the 12th International Congress on Mathematics Education: Topic Study Group 30, pp. 62236229*. Seoul: Korea National University of Education.

MEASURING AND OPTIMIZING MARKET RISK USING VINE COPULA SIMULATION

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Abstract. Copula models have increasingly become popular for the modeling of the dependence structure of financial risks. The most recent copula topic is vine copula or pair copula construction (PCC). This paper is concerned with the application of pair-copula construction for measuring and optimizing the market risk of a portfolio. The dependence among the assets is modeled using a copula based on pair-copula constructions or known as vine copula, C-vine and D-vine. Furthermore, a pairwise copula model is also discussed in this paper. In order to achieve this aim, the return of each stock price is characterized individually. The main objective of this paper is firstly to show how PCC may be useful measuring and modeling market risk. The paper are also combining important results published recently. Secondly, to show PCC can be used to model the tail dependence of log-return distribution. Thirdly, to show the uses of stepwise semiparametric estimators for the estimations of market risk in multivariate log-return data. Fourthly, to show how PCC may be used on a daily basis in finance, in particular for constructing efficient frontiers and computing Conditional VaR. The paper is also to verify whether and how the optimal portfolio composition discussed may change utilizing various types of copula families and their construction, such as pairwise or vine. To do this, four Indonesian Blue Chip stocks: ASSI=Astra International Tbk, BMRI=Bank Mandiri (Persero) Tbk., PTBA=Tambang Batubara Bukit Asam Tbk., INTP=Indocement Tunggal Perkasa Tbk recorded during the period of 6-11-2006 to 2-08-2013 are used to compose the stock portfolio. The VaR and Conditional VaR of the stock portfolio is minimized by various types of copula and their pair constructions. The use of stepwise semiparametric estimation is also demonstrated in this paper.

Key words and Phrases: Value at Risk, Conditional Valur at Risk, Vine Copula, multivariate copula

1. Introduction

In the last 10 years copula modeling has become a frequently used tool in financial market analysis. Copula modeling in multivariate distribution was initiated by Joe [10] in 1997. The copula theory available in [10] is based on hierarchical model and has been used in some new developments in multivariate modeling. The idea of Joe [10] originally proposed the pair-copula construction

(PCC) is further developed by Bedford and Cooke [4, 5], [8] and Kurowicka and Cooke [11]. Aas et al. [1] have given the inferential methodology of how to infer the parameters available in PCC. This inferential method has stimulated the use of the PCC in various applications (see, for example, Schepsmeier et al. [20], Nikoloulopoulos et al. [18], Min and Czado [14], and Mendes et al. [12]. There have also been some recent applications of copulas in the context of time series models (see the survey by Patton [19] and the recently developed COPAR model of Brechmann and Czado [7], which provides a vector autoregressive VAR model for analyzing the non-linear and asymmetric co-dependencies between two series).

This paper emphasizes the uses of vine copulae as they are useful tools to be implemented efficiently in simulating the multivariate distribution. In fact, copula functions can be used to model the dependence structure independently of the marginal distributions. By this method, a multi-variate distribution with different margins and a dependence structure can be constructed. This paper is intended to provide some applications of pair copula constructions in portfolio risk computations and efficient frontiers. The estimation of the parameters is based on the maximum likelihood method given by Vogiatzoglou [23].

Furthermore, this paper also applies the supplement (or alternative) to VaR, that is the Conditional Value-at-Risk. The CVaR risk measure is closely related to VaR. For continuous distributions, CVaR is defined as the conditional expected loss under the condition that it exceeds VaR, see Rockafellar and Uryasev [21, 22]. For continuous distributions, this risk measure also is known as Mean Excess Loss, Mean Shortfall, or Tail Value-at- Risk. However, for general distributions, including discrete distributions, CVaR is defined as the weighted average of VaR and losses strictly exceeding VaR. Recently, the redefinition of expected shortfall similarly to CVaR can be found in [2]. For general distributions, CVaR, which is a quite similar to VaR measure of risk has more attractive properties than VaR. CVaR is sub-additive and convex explained ([21] and [22]. Moreover, CVaR is a coherent measure of risk discussed in [3] and proved in [15].

The main objective of this paper is firstly to show how PCC may be useful measuring and modeling market risk. The paper are also combining important results published recently such as [1],[4, 5], [8], [11], [12], [14], and [18]. Secondly, to show PCC can be used to model the tail dependence of log-return distribution. Thirdly, to show the uses of stepwise semiparametric estimators for the estimations of market risk in multivariate log-return data. Fourthly, to show how PCC may be used on a daily basis in finance, in particular for constructing efficient frontiers and computing Conditional VaR.

2. Pair-Copulas Constructions: A brief review

Consider a stationary d dimensional process $\mathbf{X} = (X_1, \cdots, X_d)$. If $\mathbf{X}$ is a continuous random vector with joint cumulative distribution function (c.d.f.) F with density function f, and marginal c.d.f.s F_i with density functions f_i, for $i = 1, \cdots, d$, then there exists a unique copula C defined on $[0,1]^d$ such that

$$C(F_1(x_1), \cdots, F_d(x_d)) = F(x_1, \cdots, x_d) \tag{1}$$

holds for any $(x_1,...,x_d) \in \mathrm{R}^d$ (Sklar's theorem, Sklar (1959), Nelsen [17]). Therefore a copula is a multivariate distribution with standard uniform margins. Multivariate modeling through copulas allows for factoring the joint distribution into its marginal univariate distributions and a dependence structure, its copula. By taking partial derivatives of (1) one obtains

$$f(x_1, \dots, x_d; \alpha, \theta) = c_{1 \cdots d}\left(F_1(x_1), \dots, F_d(x_d)\right) \prod_{i=1}^{d} f_i(x_i; \alpha_i) \quad (2)$$

for some d-dimensional copula density $c_{1 \cdots d}$ with parameter θ.

This decomposition allows for estimating the marginal distributions f_i separated from the dependence structure given by the d-variate copula. In practice, this fact simplifies both the specification of the multivariate distribution and its estimation.

The decomposition of a multivariate distribution in a cascade of pair-copulas was originally proposed by Joe [10], and later discussed in detail by Bedford and Cooke [4],[5], Kurowicka and Cooke [11] and Aas et al.[1]. Refers to their results, Equation (2) can be represented as

$$f(x_1, \cdots, x_d) = f_d(x_d) \cdot f(x_{d-1}|x_d) \cdot f(x_{d-2}|x_{d-1}, x_d) \cdots f(x_1|x_2, \cdots, x_d). \quad (3)$$

The conditional densities in (3) can be written as functions of the corresponding copula densities. That is, for every j

$$f(x|v_1, v_2, \cdots, v_d) = c_{xvj}|\mathbf{v}j(F(x|\mathbf{v}_j), F(x|\mathbf{v}_j)), \cdots, f(x|\mathbf{v}_j), \quad (4)$$

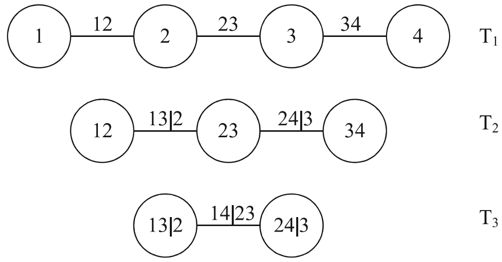
where $\mathbf{v}_j$ the d-dimensional vector $\mathbf{v}$ excluding the j-th component. For example when d = 4, then the four-dimensional canonical vine (C-vine) structure is generally expressed as

$$\begin{aligned} f(x_1, x_2, x_3, x_4; \alpha, \theta) = {} & f(x_1; \alpha_1) \cdot f(x_2; \alpha_2) \cdot f(x_3; \alpha_3) \cdot f(x_4; \alpha_4) \\ & \cdot c_{12}(F(x_1), F(x_2); \theta_{12}) \cdot c_{13}(F(x_1), F(x_3); \theta_{13}) \\ & \cdot c_{14}(F(x_1), F(x_4); \theta_{14}) \cdot c_{23}|1(F(x_2|x_1), F(x_3|x_1); \theta_{23}|1) \\ & . c_{34|}1(F(x_2|x_1), F(x_4|x_1); \theta_{34|}1) \\ & \cdot c_{34}|12(F(x_3|x_1, x_2), F(x_4|x_1, x_2); \theta_{34}|12), \end{aligned} \quad (5)$$

and the D-vine structure as

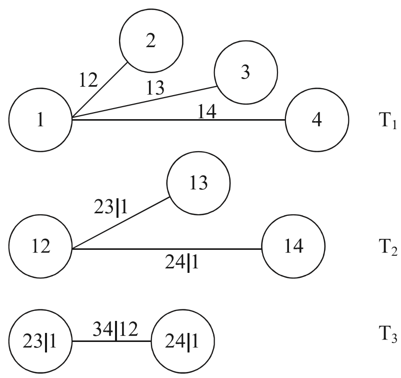
$$\begin{aligned} f(x_1, x_2, x_3, x_4; \alpha, \theta) = {} & f(x_1; \alpha_1) \cdot f(x_2; \alpha_2) \cdot f(x_3; \alpha_1) \cdot f(x_4; \alpha_1) \\ & \cdot c_{12}(F(x_1), F(x_2); \theta_{12}) \cdot c_{23}(F(x_2), F(x_3); \theta_{13}) \\ & \cdot c_{34}(F(x_3), F(x_4); \theta_{34}) \cdot c_{13}|2(F(x_1|x_2), F(x_3|x_2); \theta_{13}|2) \\ & \cdot c_{24|}3(F(x_2|x_3), F(x_4|x_3); \theta_{24|}3) \\ & \cdot c_{14}|23(F(x_1|x_2, x_3), F(x_4|x_2, x_3); \theta_{14}|23). \end{aligned} \quad (6)$$

where α and θ are the parameters of the margins and copula, respectively. The general form of the joint probability density function (d-dimensional pdf), $f()$ is



D-Vine

Figure 1. 4-dimensional D-vine copula



C-Vine

Figure 2. 4-dimensional C-vine copula

equal to

$$\prod_{k=1}^{d} f(x_k, \alpha_k) \prod_{j=1}^{d-1} \prod_{i=1}^{d-j} c_{i,i+j|i+1,\dots,i+j-1}$$

$$(F(x_i|x_{i+1},\cdots,x_{i+j-1}),F(x_{i+1}|x_{i+1},\cdots,x_{i+j-1});\theta).$$

In a D-vine there $d-1$ hierarchical tress with increasing conditioning sets, and there are $d(d-1)/2$ bivariate copulas. For a detailed description, see Asa [1]. Figure 1 shows the D-vine decomposition for $d = 4$. It consists of 3 nested trees, where T_j possess $5 - j$ nodes and $4 - j$ edges corresponding to pair copula. Figure 2 shows the C-vine decomposition for $d = 4$. It consists of 3 nested trees, where T_j possess $5 - j$ nodes and $4 - j$ edges corresponding to pair copula.

As is mention in Berg and Aas [6] and Bedford and Cooke [4], the use of vine copula is very flexible, in this case, one may choose different copula in the bivariate.

For example, one may combine the following types of (bivariate) copulas: Gaussian pair copula (no tail dependence, elliptical); t-student pair copula, Clayton (lower tail) pair copula. See Joe [10] for a copula catalogue. Here below is the example D-vine consisting of Gaussian pair-copulae and margins. Parameters, $\theta_{12}, \theta_{23}, \theta_{34}$, are estimated using maximum likelihood estimation method as in Vogiatzoglou [23]. One may refer to Haff [9] for stepwise semiparametric estimations (SSP).

$$\begin{aligned} f_{1234}(x_1, x_2, x_3, x_4; \alpha, \theta) &= c(u_1, u_2; \theta_{12}) \cdot c(u_2, u_3; \theta_{23}) \cdot c(u_3, u_4; \theta_{34}) \\ &\cdot c(u_{1|2}, u_{3|2}; \theta_{13|2}) \cdot c(u_{2|3}, u_{4|3}; \theta_{24|3}) \\ &\cdot c(u_{13|2}, u_{24|3}; \theta_{14|23}) \\ &\cdot \prod_{i=1}^{4} f(x_i; \alpha_i) \end{aligned} \tag{7}$$

In this case, θ_{13}, θ_{14}, and θ_{24} are estimated using Kendall's τ formula, that is

$$\theta_{13} = \theta_{12}\theta_{23} + \sqrt{(1-\theta_{12}^2)(1-\theta_{23}^2)}\ \sin\left(\frac{\pi\ \theta_{13|2}}{2}\right) \tag{8}$$

$$\theta_{24} = \theta_{23}\theta_{34} + \sqrt{(1-\theta_{23}^2)(1-\theta_{34}^2)}\ \sin\left(\frac{\pi\ \theta_{24|3}}{2}\right) \tag{9}$$

$$\theta_{14} = \theta_{13|2}\theta_{24|3} + \sqrt{(1-\theta_{13|2}^2)(1-\theta_{24|3}^2)}\ \sin\left(\frac{\pi \theta_{14|23}}{2}\right) \tag{10}$$

The following is the four dimensional C-vine consisting student's-*t* margins and Student's *t* pair-copulae. Another method to calculate c_{23}, c_{24} and c_{34} is to employ the stepwise semiparametric (SSP) estimators developed by Haff [9]. The following example is C-Vine copula for which the parameters are estimated using SSP method.

$$\begin{aligned} f_{1234}(x_1, x_2, x_3, x_4; \nu_M, \rho, \nu_C) &= c(u_1, u_2; \rho_{12}, \nu_{12}) \cdot c(u_1, u_3; \rho_{13}, \nu_{13}) \\ &\cdot c(u_1, u_4; \rho_{14}, \nu_{14}) \cdot c(u_{2|1}, u_{3|1}; \rho_{23|1}, \nu_{23|1}) \\ &\cdot c(u_{2|1}, u_{4|1}; \rho_{24|1}, \nu_{24|1}) \\ &\cdot c(u_{23|1}, u_{24|1}; \rho_{34|12}, \nu_{34|12}) \\ &\cdot \prod_{i=1}^{4} f(x_i; \nu_i) \end{aligned} \tag{11}$$

where

$$f(x; \nu) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\pi\nu}(\nu/2)} \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2} \tag{12}$$

$$
\begin{aligned}
c(u_i,u_j;\rho,\nu) &= \Gamma\left(\frac{\nu+2}{2}\right)\Gamma\left(\frac{\nu}{2}\right)\left(1+\frac{t_\nu^{-1}(u_i)^2}{\nu}\right)^{(\nu+1)/2}\left(1+\frac{t_\nu^{-1}(u_j)^2}{\nu}\right)^{(\nu+1)/2} \\
&\quad \Big/ \left(\frac{\nu+1}{2}\right)^2\sqrt{1-\rho^2} \\
&\quad \times\left(1+\frac{t_\nu^{-1}(u_i)^2+t_\nu^{-1}(u_j)^2-2\rho t_\nu^{-1}(u_i)t_\nu^{-1}(u_j)}{\nu(1-\rho^2)}\right) \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
u_{2|1} &= h(u_1,u_2;\rho_{12},\nu_{12}), \\
u_{3|1} &= h(u_1,u_3;\rho_{13},\nu_{13}), \\
u_{4|1} &= h(u_1,u_4;\rho_{14},\nu_{14}), \\
u_{23|1} &= h(u_{2|1},u_{3|1};\rho_{23|1},\nu_{23|1}), \\
u_{24|1} &= h(u_{2|1},u_{4|1};\rho_{24|1},\nu_{24|1}), \\
u_{34|12} &= h(u_{23|1},u_{24|1};\rho_{34|12},\nu_{34|12}), \\
h(u_i,u_j;\rho,\nu) &= t_{\nu+1}(\sqrt{\nu+1}(t_\nu^{-1}(u_i)-\rho t_\nu^{-1}(u_j))((\nu+t_\nu^{-1}(u_j)^2)(1-\rho^2))^{-1/2}) \qquad (14)
\end{aligned}
$$

where $u_i = t_v(x_i)$, and t_v is the c.d.f. of the Students t-distribution with v degrees of freedom. See Haff [9] for d-dimensional C-vine formulation. As reported by Vogiatzoglou in [23] that the log likelihood function of the canonical vine t-copula model (tCVine) did not converge to a solution, hence SSP reported by Haff in [9] is alternative estimation methods. Mention in by Haff [9] that SSP estimator is computationally tractable even in high dimensions, as opposed to other methods such as MLE or IFM method.

3. Optimization of Value at Risk

Consider a portfolio consist of d stocks and denote by w_i the weight of stock i allocated to the portfolio at time t. Let $f(\mathbf{w},\mathbf{r})$ be the loss function of the portfolio, with $\mathbf{w} \in R^d$ is the vector of the portfolio and $\mathbf{r}$ is the vector of asset returns. Let ξ be a certain threshold, the Value-at-Risk of a portfolio at level α is defined as the lower $\alpha-$quantile of the distribution of the portfolio return

$$\mathrm{VaR}(\alpha) = \inf\{\xi \in R : P(f(\mathbf{w},\mathbf{r}) \le \xi) \ge \alpha\} \qquad (15)$$

The uses of VaR to measure financial risks have been increasingly popular since 1990s. VaR now becomes the standard risk measure used by financial analysts to quantify the market risk of an asset or a portfolio.

CVaR is a supplement or an alternative to VaR. CVaR is another percentile risk measure which is called Conditional Value-at-Risk. For continuous distributions, CVaR is defined as the conditional expected loss under the condition that it exceeds VaR. The following approach to CVaR is summarized from [21] and [22]. Let r_p be the return of the portfolio, then r_p is defined as a random variable satisfying $r_p = w_1r_1 + w_2r_2 {}^{+} \cdots + w_dr_d = \mathbf{w}^T\mathbf{r}$ and the weight constrain condition is imposed to be $\sum_{i=1}^{d} w_i = 1$ If the short position is not allowed then $w_i \ge 0$ for all $t = 1,\cdots,d$. Let $p(\mathbf{r})$ be the joint distribution of the uncertain return of the assets, the probability of r_p exceeding a certain amount r^* is given by

$$\int_{r_p > r^*} p(r)d(r) \text{ then is equivalent to } \Psi(w,\xi) = \int_{f(w,r)\le\xi} p(r)d(r) \tag{16}$$

where $\Psi(\mathbf{w},\xi)$ represents the cumulative distribution function for the associated loss w. Assuming $\Psi(\mathbf{w},\xi)$ is continuous with respect to ξ, VaR(α) and CVaR(α) for the loss $f(\mathbf{w},\mathbf{r})$ associated with $\mathbf{w}$ any probability level $\alpha \in (0,1)$ can be defined by

$$\text{VaR}(\alpha,w) = \min\{\xi \in R{:}\, \Psi(w,\xi \ge \alpha\} \tag{17}$$

$$\text{CVaR}(\alpha,w) = \frac{1}{1-\alpha}\int_{r\in R^n} \max\{[f(w,r) - \xi, 0]p(r)d(r) \tag{18}$$

Equation (17) is read as the smallest value ξ such that the probability $P[f(\mathbf{w},\mathbf{r}) > \xi]$ of a loss exceeding ξ is not larger that $1 - \alpha$. Therefore, VaR_α presents $(1 - \alpha)$-quantile of the loss distribution $\Psi(\mathbf{w},\xi)$ and CVaR_α presents the conditional expected loss associated with $\mathbf{w}$ if VaR_α is exceeded. Following [22], the CVaR(α) of the loss associated with any $\mathbf{w}$, it is found

$$\text{CVaR}(\alpha) = \min_{\xi\in\mathrm{R}} F_\alpha(\mathbf{w},\xi), \tag{19}$$

with

$$F_\alpha(\mathbf{w},\xi) = \xi + \frac{1}{1-\alpha)}\int_{\mathbf{r}\in\mathbb{R}^n} \max[f(\mathbf{w},\mathbf{r}) - \xi, 0]\; p(\mathbf{r})\; d\mathbf{r} \tag{20}$$

Now, the conditional value-at-risk, CVaR, is defined as the solution of an optimization problem

$$\text{CVaR}(\alpha,\mathbf{w}) = \xi + \frac{1}{1-\alpha}\int_{\mathbf{r}\in\mathbb{R}^n} \max\{(f(\mathbf{w},\mathbf{r}) - \xi), 0\}\; p(\mathbf{r})\; d\mathbf{r} \tag{21}$$

where α is the probability level such that $0 < \alpha < 1$. CVaR also is known as Mean Excess Loss, Mean Shortfall (Expected Shortfall), or Tail Value-at-Risk, see [2]. However, for general distributions, including discrete distributions, CVaR is defined as the weighted average of VaR and losses strictly exceeding VaR, see [21]. Some properties of CVaR and VaR and their relations are studied in [2] and [15]. For general distributions, CVaR, which is a quite similar to VaR measure of risk has more attractive properties than VaR. CVaR is sub-additive and convex see [21]. Moreover, CVaR is a coherent measure of risk, proved first in [15], see also [2] and
[21].

Therefore, the problem of minimizing the Conditional Value at Risk can thus be formulated as the following:

$$\text{minimize} \quad \xi + \frac{1}{1-\alpha}\int_{\mathbf{r}\in\mathbb{R}^n} \max\{(f(\mathbf{w},\mathbf{r}) - \xi), 0\}\; p(\mathbf{r})\; d\mathbf{r} \tag{22}$$

$$\text{subject to} \quad \sum_{i=1}^{n} w_i = 1,\; x_i \ge 0 \tag{23}$$

$$-\mathbf{w}^T E(\mathbf{r}) \le -r^* \tag{24}$$

Table 1. Descriptive Statistics

Statistics	ASII	BMRI	PTBA	INTP
Mean	0.0009	0.0006	0.0006	0.0009
Standard deviation	0.0276	0.0271	0.0317	0.0307
Skewness	0.1306	0.3690	-0.2404	0.370
Kurtosis	10.0123	8.2807	13.0502	45.0897
Jarque-Bera Test (5%)	1	1	1	1
1 = reject H_0				
Jarque-Bera Statistics	35700	20612	73398	12848
Jarque-Bera *p*-Values	0.001	0.001	0.001	0.001
Jarque-Bera Crit-Value	5.9586	5.9586	5.9586	5.9586

In this case the feasibility set is X defined on region satisfying (23) and (24). Set X is convex (it is polyhedral, due to linearity in constraint (23) and (24). The optimization problem (22)-(24) are a convex programming. This problem is solved using *PortfolioCVaR class* of MATLAB R2013a.

4. Empirical studies

In this section, the models of stock portfolio risk measurement and management (described in the previous sections) are implemented to a hypothetical portfolio composed by 4 Indonesian Blue Chip stocks: ASSI=Astra International Tbk, BMRI=Bank Mandiri (Persero) Tbk., PTBA=Tambang Batubara Bukit Asam Tbk., INTP=Indocement Tunggal Perkasa Tbk. The historical data are recorded during the period of 6 November 2006 to 2 August 2013. Precisely, the statistics of the 4 stocks are described in Table 1.

Table 1 reflects the mean, standard deviation, skewness, and kurtosis of the daily returns of four stocks over the period from 6 November 2006 to 2 August 2013. The kurtosis of all stock returns exceeds the kurtosis of normal distribution (3.0) substantially. The Jarque-Bera tests of the null hypothesis that the return distribution follow a normal distribution against the alternative that the return do not come from a normal distribution. As seen from Table 1. that the test statistics exceed the critical value at the 5% level of significant, this means that all stock returns are not normally distributed, it shows a fat tailed distribution. This means that the probability of extreme events is higher than the probability of extreme events under the normal distribution. Therefore, capturing the stock returns using normal distribution could be underestimated. When the skewness of the return data are considered, it shows that all returns are right skewed except for PTBA. This suggests that the returns are not symmetry. Again, capturing by normal distribution may result in misleading conclusions.

Using results in Table 2 and Table 3, the value of c_{13}, c_{14}, and c_{24} can be calculated using Equation (8)-(10), giving the correlation matrix ρ. Using ρ, the

Table 2. Estimated parameters and the (standard errors) from D-vine models

D-vine copula						
	Clayton		SJC			
c_{12}	0.3057	(0.012)	0.3374	(0.036)	0.2270	(0.038)
c_{23}	0.2636	(0.014)	0.1480	(0.018)	0.0541	(0.038)
c_{34}	0.1810	(0.015)	0.0992	(0.032)	0.0749	(0.033)
$c_{13\|2}$	0.1059	(0.015)	0.4167	(0.017)	0.3650	(0.027)
$c_{24\|3}$	0.1793	(0.015)	0.2269	(0.027)	0.0863	(0.028)
$c_{14\|23}$	0.0982	(0.014)	0.2212	(0.024)	0.0538	(0.029)
AIC	-1483.7095		-1871.350			
BIC	-1450.9396		-1805.8104			
$\mathcal{LL}$	747.855		948.753			

Table 3. Estimated parameters and the (standard errors) from C-vine models

D-vine copula						
	Clayton		SJC			
c_{12}	0.3057	(0.012)	0.3374	(0.036)	0.2270	(0.038)
C_{13}	0.2337	(0.014)	0.2274	(0.031)	0.0704	(0.031)
C_{14}	0.2401	(0.014)	0.0415	(0.027)	0.0266	(0.023)
$C_{23\|1}$	0.1542	(0.015)	0.4127	(0.024)	0.3112	(0.030)
$c_{24\|1}$	0.1391	(0.016)	0.3133	(0.029)	0.1864	(0.033)
$c_{14\|2}$	0.0464	(0.014)	0.1604	(0.034)	0.0005	(0.003)
AIC	-1487.152		-1873.5069			
BIC	-1454.382		-1807.9672			
$\mathcal{LL}$	749.576		947.675			

return of the portfolio (r_p) consisting of 4 assets are simulated by Gaussian copularnd() of MatLab function. The vector weight $\mathbf{w}$ = [0.25 0.25 0.25 0.25] is used to compose the portfolio. Then apply *PortfolioCVaR class* [16] to $r_p = \mathbf{w}^T\mathbf{r}$. The results are summarized in Table 4. Note that, for *t* C-Vine model, we use stepwise semiparametric estimator (SSP) proposed by Haff [9] (Eqn.(11)-(14)), to construct correlation matrix ρ.

Table 4 reflects the value of VaR and CVaR simulated by Clayton D-Vine, *t* C-Vine, Symmetrized Joe-Clayton (SJC) D-Vine, Gaussian pairwise, *t* student pairwise, and multivariate normal at 1% and 5% significant levels. The multivariate normal calculated using the standard Markowitz mean-variance (MV) framework. As it was shown in [22] that when the loss functions come from normal distribution then VaR and CVaR are equivalent in the sense that they generate the same efficient frontier. However, in the case of nonnormal distribution, and especially skewed distributions, CVaR and MV portfolio optimization approaches result in significant

Table 4. Simulated VaR and CVaR

Methods	α = 0.01		α = 0.05	
	VaR (%)	CVaR (%)	VaR (%)	CVaR (%)
Clayton D-Vine	4.99	5.99	3.07	4.32
t C-Vine	6.69	9.59	3.50	5.57
SJC D-Vine	4.75	6.23	2.83	4.08
Gaussian pairwise	5.68	7.75	3.33	4.81
t student pairwise	5.76	8.07	3.19	5.05
Multivariate Normal	5.01	5.77	3.71	4.63

differences. The main idea of using the CVaR optimization technique is that it can reshape one tail of the loss distribution, which corresponds to high losses, and it can not account for the opposite tail representing high profits. On the other hand, the MV approach consider the risk as the variance of the loss distribution, and since the variances (risk) can come from both tails (upper and lower), hence it is affected by high gains as well as by high losses.

5. Concluding remark

In this paper we have explored the potentials of PCC to model the dependence among financial data. A fully flexible multivariate distribution was obtained by constructing marginals with different distribution into C-vine and D-vines model. The marginals of the data show the asymmetric, high kurtosis, and skewed right except for PTBA. The simulation results show that *t*-student C-vine gives highest estimation compare with the other model (see Table 4). This is due to the fact that the parameters of *t*-student C-vine model is estimated by stepwise semiparametric estimation, which is according to Haff [9] giving higher estimates for correlations.

The pair copulas construction still needs further research on tests for choosing among copula families and among decompositions, and more powerful goodness-offit tests. Further research topics include time-varying pair-copulas may be one of interest in multivariate modeling.

Figure 3 shows that for a given risk (volatility), one would like to choose a portfolio that gives you the greatest possible rate of return, in this case one may choose Clayton CVine. Gaussian Pairwise gives the lowest rate of return for a given risk.

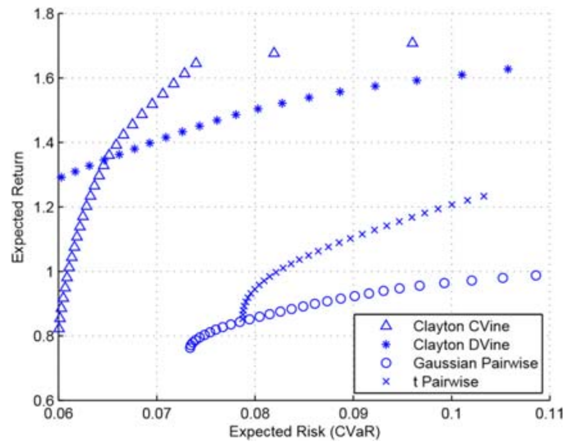


Figure 3. Efficient frontiers simulated by pairwise and vine copula with $\alpha = 1\%$

References

[1] Aas, K., Czado, C., Frignessi, A. and Bakken, H. *Pair-copula constructions of multiple dependence*. Insurance: Mathematics Economics. (2009) 44:128-198

[2] Acerbi, C., Nordio, C., Sirtori, C. *Expected Shortfall as a Tool for Financial Risk Management*. Working Paper, can be downloaded from: http://gloriamundi.com/Library Journal.asp

[3] Artzner, P., Delbaen F., Eber, J. M. and D. Heath (1999), *Coherent Measures of Risk*. Mathematical Finance, 9, 203-228.

[4] Bedford, T. and Cooke, R. M. *Probability density decomposition for conditionally dependent random variables modeled by vines*. Annals of Mathematical and Artificial Intelligence (2001) 32: 245.268.

[5] Bedford, T. and Cooke, R. M. *Vines: a new graphical model for dependent random variables*. Ann. Statist. (2002). 30(4): 1031.1068.

[6] Berg, D. and Aas, K. *Models for construction of higher-dimensional dependence: A comparison study*. European Journal of Finance (2009), 15, 639-659.

[7] Brechmann, E.C., and C. Czado. *COPAR - multivariate time-series modelling using the COPula AutoRegressive model*, Working Paper, Faculty of Mathematics, Technical University of Munich. (2012)

[8] Cooke, R.M., H. Joe and K. Aas, *Vines Arise, chapter 3 in DEPENDENCE MODELING Vine Copula Handbook*, Ed. D. Kurowicka and H. Joe, World Scienti c Publishing Co, Singapore. (2011)

[9] Haff, I. H. *Parameter estimation for pair-copula constructions*. Bernoulli 19(2), (2013), 462491

[10] Joe, H. *Multivariate Models and Dependence Concepts*. (1997). London: Chapman & Hall

[11] Kurowicka, M. and Cooke, R. M. *Uncertainty Analysis with High Dimensional Dependence Modelling* (2006), John Wiley & Sons, New York.

[12] Mendes, B.V.d.M., Semeraro, M.M., Leal, R.P.C., *Pair-copulas modeling in finance*. Financial Markets and Portfolio Management (2010), 24 (2), 193-213.

[13] Min, A., Czado, C., *Bayesian inference for multivariate copulas using pair-copula constructions*. Journal of Financial Econometrics,(2010), 8(4).

[14] Min, A., Czado, C., *Bayesian model selection for multivariate copulas using pair-copula constructions*. Canadian Journal of Statistics (2011), 39(2), 239-258.

[15] Pflug G. Ch. *Some Remarks on the Val-ue-At-Risk and the Conditional Value-At-Risk* In: S. Uryasev, Ed., Pro- babilistic Constrained Optimization: Methodology and Applications, Kluwer Academic Publishers, Dordrecht, (2000)

[16] MATLAB version 8.01.0 (R2013a). Natick, Massachusetts: The MathWorks Inc., (2013).

[17] Nelsen R. B. *An Introduction to Copulas*. Second Editions. Springer Science+Business Media, Inc. (2006)

[18] Nikoloulopoulos, A. K., Joe, H., and Li, H. . Vine copulas with asymmetric tail dependence and applications to financial return data. Computational Statistics & Data Analysis (2012), 56:3659-3673.

[19] Patton A. J. *A review of copula models for economic time series*. Journal of Multivariate Analysis 110 (2012) 418

[20] Schepsmeier U., J.Stoeber, and E.C. Brechmann. (2013). Statistical inference of vine copulas: Vine Copula Package Version February 15, 2013. http://cran.rproject.org/web/packages/VineCopula/VineCopula.pdf

[21] Rockafellar R. T. and S. Uryasev. *Optimization of Conditional Value-at-Risk*. Journal of Risk, Vol. 2, (2000), pp. 21-41

[22] Rockafellar, R. T. and S. Uryasev, *Conditional Value-at-Risk for General Loss Distributions*, Journal of Banking and Finance (2002), Vol. 26, pp. 1443-1471.

[23] Vogiatzoglou, M. *Dynamic Copula Toolbox*, University of Macedonia, Egnatias 156, Thessaloniki, Greece. Version: November 07, (2010)

MONTE CARLO AND MOMENT ESTIMATION FOR PARAMETERS OF A BLACK SCHOLES MODEL FROM AN INFORMATION-BASED PERSPECTIVE (THE BS-BHM MODEL):A COMPARISON WITH THE BS-BHM UPDATED MODEL

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Abstract. This paper presents estimation of parameters on the BS-BHM model by using Monte Carlo and Moments estimate as they have been done in BS-BHM Updated model. BS-BHM Updated model is BS-BHM model that it is improved the result of Gaussian integral, especially in completing square. Estimation of parameters use Monte Carlo and moments estimate under BS-BHM model results the equation of polynomial of four degree . While estimation of parameters under BS-BHM Updated model results the quadratic equation. Application for real data of Microsoft shares (MSFT), under BS-BHM model results four different estimates values, while under BS-BHM Updated model results one estimate value.

Key words and Phrases : BS-BHM model, Monte Carlo estimate, Moment estimate,and Comparison.

1. Introduction

Black Scholes asset pricing model from an information-based perspective has been developed by Brody Hughston Macrina (BHM). It is developed from a special condition of asset pricing model from an information-based approaches by Brody Hughston Macrina (BHM model or BHM approach). Further, Black Scholes model from an information-based perspective is called BS-BHM model in this paper. Explicitly, BS-BHM model is presented, see Macrina, A.[6]

$$S_t = P_{tT}\, S_0 \exp\left(rT - \frac{1}{2}\nu^2 T + \frac{1}{2}\frac{\nu\sqrt{T}}{\sigma^2\tau + 1} + \frac{\sigma\tau\,\nu\sqrt{T}}{t(\sigma^2\tau + 1)}\,\xi_t\right) \tag{1.1}$$

or

$$S_t = S_0 \exp\left(rt - \frac{1}{2}\nu^2 T + \frac{1}{2}\frac{\nu\sqrt{T}}{\sigma^2\tau + 1} + \frac{\sigma\tau\,\nu\sqrt{T}}{t(\sigma^2\tau + 1)}\,\xi_t\right) \tag{1.2}$$

or

$$\frac{S_t}{S_0} = \exp\left(rt - \frac{1}{2}v^2T + \frac{1}{2}\frac{v\sqrt{T}}{\sigma^2\tau + 1} + \frac{\sigma\tau\, v\sqrt{T}}{t(\sigma^2\tau + 1)}\,\xi_t\right) \tag{1.3}$$

The BS-BHM model in equation (1.2) which it is equal to equation (1.3) has two parameters i.e. the asset price volatility parameter ν and the information flow rate parameter σ. It can be showed that the random variable of $\frac{S_t}{S_0}$ has lognormal distribution with density function

$$f\left(\frac{S_t}{S_0}\right) = \frac{1}{\frac{S_t}{S_0}\sqrt{2\pi}\; b\sqrt{\sigma^2t^2 + \frac{t(T-t)}{T}}}\exp\left\{-\frac{1}{2}\frac{\left(\log\frac{S_t}{S_0} - A\right)^2}{b^2\left(\sigma^2t^2 + \frac{t(T-t)}{T}\right)}\right\} \tag{1.4}$$

or

$$f\left(\frac{S_t}{S_0}\right) = \frac{1}{\frac{S_t}{S_0}\sqrt{2\pi}\; B}\exp\left\{-\frac{1}{2}\frac{\left(\log\frac{S_t}{S_0} - A\right)^2}{B^2}\right\} \tag{1.5}$$

where $A = rt - \frac{1}{2}v^2T + \frac{1}{2}\frac{v\sqrt{T}}{\sigma^2\tau + 1}$ and

$$B^2 = b^2\left(\sigma^2t^2 + \frac{t(T-t)}{T}\right) = \left(\frac{\sigma\tau\, v\sqrt{T}}{t(\sigma^2\tau + 1)}\right)^2\left(\sigma^2t^2 + \frac{t(T-t)}{T}\right)$$

In other words, random variable of $\log\frac{S_t}{S_0}$ is normally distributed with mean is $A = rt - \frac{1}{2}v^2T + \frac{1}{2}\frac{v\sqrt{T}}{\sigma^2\tau + 1}$ and variance is $B^2 = \left(\frac{\sigma\tau\, v\sqrt{T}}{t(\sigma^2\tau + 1)}\right)^2\left(\sigma^2t^2 + \frac{t(T-t)}{T}\right)$. Further, it is written by $\log\frac{S_t}{S_0} \sim N(A, B^2)$.

The BS-BHM model as in equation (1.2) is derived from BHM model by a special condition. BHM model is built for case of cash flow is payout of the associated dividend of equity. Further, the process of deriving of BS-BHM model as in Macrina,A [6] and Mutijah, Guritno, S. and Gunardi [7].

Explicitly, the asset pricing model is presented as follows,

$$S_t = P_{tT}E^{\mathbb{Q}}[D_T|\mathcal{F}_t] \tag{1.6}$$

S_t is the value of cash flows at time t, $0 \leq t < T$ from asset that payout single dividend D_T at time T . In equation (1.6), P_{tT} represents the discount factors that it is to be equal to $e^{-r(T-t)}$ with r is the interest rates. Then $\mathbb{Q}$ is the risk neutral probability, and $\mathcal{F}_t$ is the market information filtration.

Modeling the infornation flows is based on an assumption that the information about dividends which is available in market is contained by the process $\{\xi_t\}_{0\leq t\leq T}$ defined by :

$$\xi_t = \sigma t D_T + \beta_{tT} \tag{1.7}$$

$\{\xi_t\}$ is a market information process. The market information process is composed from two parts, they are σtD_T which refers to the true information about dividends and $\{\beta_{tT}\}_{0\leq t\leq T}$ which refers to a standard Brownian Bridge on interval [0, T]. In the formula of asset pricing model by Brody Hughston Macrina in equation (1.6) above, if random variable D_T is equal to x which it has continuous distribution then ,

$$E^{\mathbb{Q}}[D_T|\mathcal{F}_t] = E^{\mathbb{Q}}[D_T|\,\xi_t] = \int_0^\infty x\pi_t(x)\,dx \tag{1.8}$$

where

$$\pi_t(x) = \frac{d}{dx}\,\mathbb{Q}(D_T \leq x|\xi_t) \tag{1.9}$$

By using Bayes formula in Box-Tiao [2], $\pi_t(x)$ is presented in Brody, D.C, Hughston, L.P, and Macrina, A [1], Caliskan, N [3], Macrina,A [6] and Mutijah, Guritno, S. and Gunardi [7] as follows

$$\pi_t(x) = \frac{p(x)\rho(\xi_t|D_T=x)}{\rho(\xi_t)} \quad (1.10)$$

and the final result of the BHM model or the BHM approach,

$$S_t = P_{tT} \frac{\int_0^\infty x\, p(x) \exp\left(\frac{T}{T-t}\left(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t\right)\right) dx}{\int_0^\infty p(x) \exp\left(\frac{T}{T-t}\left(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t\right)\right) dx} \quad (1.11)$$

Brody Hughston Macrina also built the other concept for the asset pricing model that it is derived from the formula of equation (1.6) for a special condition where it is a limited-liability asset which pays no interim dividends and at time T it is sold off for the value S_T. S_T is log-normally distributed and has the form of

$$S_T = S_0 \exp(rT - \frac{1}{2} v^2 T + v\sqrt{T} X_T) \quad (1.12)$$

where S_0, r, $\boldsymbol{v}$ are given constants and X_T is a standard normally distributed random variable. The corresponding information process is given by

$$\xi_t = \sigma t X_T + \beta_{tT} \quad (1.13)$$

The price proses $\{S_t\}_{0\le t\le T}$ is obtained from :

$$S_t = P_{tT}\, E^{\mathbb{Q}}(\Delta_T(X_T)|\, \xi_t) \quad (1.14)$$

Then for $t < T$, the equation S_t results :

$$S_t = P_{tT} \int_{-\infty}^{\infty} \Delta_T(x)\, \pi_{tT}(x)\, dx \quad (1.15)$$

And by the Bayes formula, it is obtained $\pi_{tT}(x)$ as follows

$$\pi_{tT}(x) = \frac{p(x) \exp\left[\frac{T}{T-t}\left(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t\right)\right]}{\int_{-\infty}^{\infty} p(x) \exp\left[\frac{T}{T-t}\left(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t\right)\right] dx}$$

(1.16)

In this case, S_T plays the role of single cash flow $\Delta_T(x)$ for $X_T = x$.

So, it is obtained the equation S_t as follows

$$S_t = P_{tT} \int_{-\infty}^{\infty} S_0 \exp\left(rT - \frac{1}{2} \nu^2 T + \nu\sqrt{T}\, x\right) \frac{p(x) \exp\left[\frac{T}{T-t}\left(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t\right)\right]}{\int_{-\infty}^{\infty} p(x) \exp\left[\frac{T}{T-t}\left(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t\right)\right] dx} dx \quad (1.17)$$

Because X_T is assumed to be standard normally distributed then

$$p(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2} x^2) \quad (1.18)$$

To follow the Gaussian integrals in Macrina, A [6] and Straub, W.O. [12] then S_t becomes

$$S_t = P_{tT} S_0 \exp\left(rT - \frac{1}{2} \nu^2 T\right) \frac{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-\frac{1}{2}x^2) \exp\left[\left(\frac{T}{T-t}\sigma \xi_t + \nu\sqrt{T}\right)x - \frac{1}{2}\frac{T}{T-t}\sigma^2 t x^2\right] dx}{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-\frac{1}{2}x^2) \exp\left[\frac{T}{T-t}\sigma \xi_t x - \frac{1}{2}\frac{T}{T-t}\sigma^2 t x^2\right] dx}$$

(1.19)

By using Gaussian integrals, the equation of asset pricing model S_t is given below

$$S_t = S_0 \exp\left(rt - \frac{1}{2}\frac{\sigma^2 \tau}{\sigma^2 \tau + 1}\nu^2 T + \frac{\sigma \tau\, \nu\sqrt{T}}{t(\sigma^2\tau + 1)}\, \xi_t\right)$$

(1.20)

where $\tau = \frac{tT}{(T-t)}$. Successive steps to obtain the model in equation (1.20) can be seen in Mutijah, Guritno, S. and Gunardi [7]. Furthermore, the model in equation (1.20) is called the BS-BHM Updated model.

In BS-BHM model, there are also the asset price volatility parameter ν and

true information flow rate parameter σ which they can not be observed directly. This paper will discuss the two parameter estimation for the previous behaviour of asset price. The estimation value of parameter ν and σ arisen from this general procedure is called a historical volatility and information flow rate estimation.
The deriving of BHM model with a special condition which is built by Brody Hughston Macrina , it must be as in equation (1.20).Therefore, this paper will compare the results of parameter estimation between estimation of parameters and its application for real data of Microsoft shares (MSFT) under the BS-BHM model with under the BS-BHM Updated model. Hence, estimation of parameters and its application for real data of Microsoft shares (MSFT) under the BS-BHM Updated model can be seen in Mutijah, Guritno, S. and Gunardi [8].

2. Main Results

2.1. Monte Carlo Estimate

Estimation procedure of Monte Carlo under the BS-BHM the same as estimation procedure of Monte Carlo under the BS-BHM Updated model. They are based on procedure by Higham,D.J. [5] as follows

Suppose that historical asset price data is available at equally spaced time values $t_i = i\,\Delta t$,so S_{t_i} is the asset price at time t_i . Defined $U_i = \log \frac{S_{t_i}}{S_{t_{i-1}}}$ and $\{U_i\}$ are independent as in Higham, D.J. [5]. To estimate the asset price volatility ν and the information flow rate σ on the BS-BHM model by using Monte Carlo approach that it is written Higham, D.J. [5] as follows :

Suppose that $t = t_n$ is the current time and that the M+1 is most current asset prices. $\left\{S_{t_{n-M}},\ S_{t_{n-M+1}}, \ldots\ S_{t_{n-1}}, S_{t_n}\right\}$ is also available and by using the corresponding log rasio data which is $\left\{U_{n+1-i}\right\}_{i=1}^{M}$, then the sample mean dan variance estimation are

$$a_M = \frac{1}{M} \sum_{i=1}^{M} U_{n+1-i} \tag{2.1}$$

and

$$b_M^2 = \frac{1}{M-1} \sum_{i=1}^{M} \left(U_{n+1-i} - a_M\right)^2 \tag{2.2}$$

Monte Carlo estimate is done by comparing the sample mean with the mean of BS-BHM model or by comparing the sample variance with the variance of BS-BHM model as below

$$a_M = rt - \frac{1}{2}\nu^2 T + \frac{1}{2}\frac{\nu\sqrt{T}}{\sigma^2\tau + 1}$$

or

$$\nu^2 - \frac{1}{\sqrt{T}\,(\sigma^2\tau + 1)}\nu + \frac{2\,(a_M - rt)}{T} = 0 \tag{2.3}$$

and

$$b_M^2 = \left(\frac{\sigma\tau\,\nu\sqrt{T}}{t(\sigma^2\tau + 1)}\right)^2 \left(\sigma^2 t^2 + \frac{t(T-t)}{T}\right)$$
$$= \left(\frac{\sigma\tau\sqrt{T}}{t(\sigma^2\tau + 1)}\right)^2 \left(\sigma^2 t^2 + \frac{t(T-t)}{T}\right)\nu^2$$

or

$$\nu^2 = \frac{b_M^2}{\left(\frac{\sigma\tau\sqrt{T}}{t(\sigma^2\tau + 1)}\right)^2 \left(\sigma^2 t^2 + \frac{t(T-t)}{T}\right)} \tag{2.4}$$

Substitution equation (2.4) to equation (2.3) is obtained by algebra tricks successively as below

$$\frac{b_M^2}{\left(\frac{\sigma\tau\sqrt{T}}{t(\sigma^2\tau+1)}\right)^2\left(\sigma^2t^2+\frac{t(T\text{-}t)}{T}\right)} - \frac{1}{\sqrt{T}\,(\sigma^2\tau+1)}\cdot\frac{\sqrt{b_M^2}}{\left(\frac{\sigma\tau\sqrt{T}}{t(\sigma^2\tau+1)}\right)\sqrt{\sigma^2t^2+\frac{t(T\text{-}t)}{T}}} + \frac{2\,(a_M-rt)}{T} = 0$$

Making the same in denumerator to the equation is obtained

$$\frac{b_M^2\,t^2(\sigma^2\tau+1)^2 - \sigma\tau t\sqrt{b_M^2}\,\sqrt{\sigma^2t^2+\frac{t(T\text{-}t)}{T}} + 2\,(a_M-rt)(\sigma\tau\sqrt{T})^2\left(\sigma^2t^2+\frac{t(T\text{-}t)}{T}\right)}{(\sigma\tau\sqrt{T})^2\left(\sigma^2t^2+\frac{t(T\text{-}t)}{T}\right)} = 0$$

By dividing to multiplying then

$$b_M^2\,t^2(\sigma^2\tau+1)^2 - \sigma\tau t\sqrt{b_M^2}\,\sqrt{\sigma^2t^2+\frac{t(T\text{-}t)}{T}} + 2\,(a_M-rt)(\sigma\tau\sqrt{T})^2\left(\sigma^2t^2+\frac{t(T\text{-}t)}{T}\right) = 0$$

Changing in form of equation the same as

$$\sigma\tau t\sqrt{b_M^2}\,\sqrt{\sigma^2t^2+\frac{t(T\text{-}t)}{T}} = b_M^2\,t^2(\sigma^2\tau+1)^2 + 2\,(a_M-rt)(\sigma\tau\sqrt{T})^2\left(\sigma^2t^2+\frac{t(T-t)}{T}\right)$$

And by multiplying to dividing is obtained

$$\sqrt{\sigma^2t^2+\frac{t(T\text{-}t)}{T}} = \frac{\sqrt{b_M^2}\;t\,(\sigma^2\tau+1)^2}{\sigma\tau} + \frac{2\,(a_M-rt)(\sigma\tau\sqrt{T})^2\left(\sigma^2t^2+\frac{t(T-t)}{T}\right)}{t\sqrt{b_M^2}}$$

The left and right hands are squared

$$\sigma^2t^2+\frac{t(T\text{-}t)}{T} = \frac{b_M^2\,t^2(\sigma^2\tau+1)^4}{\sigma^2\tau^2} + 4T\,(a_M-rt)\left(\sigma^2t^2+\frac{t(T-t)}{T}\right)(\sigma^2\tau+1)^2 + \frac{4\,(a_M-rt)^2(\sigma^2\tau^2T^2)\left(\sigma^2t^2+\frac{t(T-t)}{T}\right)^2}{t^2\,b_M^2}$$

Thus it is made in equation is to be equal to zero

$$\frac{b_M^2\,t^2(\sigma^2\tau+1)^4}{\sigma^2\tau^2} + 4T(a_M-rt)\left(\sigma^2t^2+\frac{t(T-t)}{T}\right)(\sigma^2\tau+1)^2 + \frac{4\,(a_M-rt)^2(\sigma^2\tau^2T^2)\left(\sigma^2t^2+\frac{t(T-t)}{T}\right)^2}{t^2\,b_M^2} - \sigma^2t^2 - \frac{t(T\text{-}t)}{T} = 0$$

Successively, by algebra tricks then it is held the final result is the polynomial equation as below

$$A\sigma^8 + B\sigma^6 + C\sigma^4 + D\sigma^2 + E = 0$$

where

$$A = b_M^2\,t^2\tau^2 + 4T\,(a_M-rt)\,t^2\,\tau^2 + \frac{4\,(a_M-rt)^2\,\tau^2T^2t^2}{b_M^2}$$

$$B = b_M^2\,t^2\tau + 4T\,(a_M-rt)\left(2t^2\tau + \frac{t\,(T-t)\,\tau^2}{T}\right) + \frac{8\,(a_M-rt)^2\,\tau^2\,t\,T(T\text{-}t)}{b_M^2}$$

$$C = 6\,b_M^2\,t^2 + 4T\,(a_M-rt)\left(t^2+\frac{t\,(T-t)\,\tau}{T}\right) + \frac{4\,(a_M-rt)^2\,\tau^2\,(T\text{-}t)^2}{b_M^2}$$

$$D = \frac{4\,b_M^2\,t^2}{\tau} + 4(a_M-rt)t(T-t) - \frac{t\,(T-t)}{T}$$

$$E = \frac{b_M^2\,t^2}{\tau^2}$$

Let $\sigma^2 = x$, then it is obtained the polynomial of four degree

$$Ax^4 + Bx^3 + Cx^2 + Dx + E = 0$$

The equation of polynomial of four degree has four solution in $x = \sigma^2$.

Further, to determine estimation of parameter ν then the solutions in $x = \sigma^2$ are substituted to equation (2.3) that is

$$\nu^2 - \frac{1}{\sqrt{T}\,(\sigma^2\tau + 1)}\nu + \frac{2\,(a_M - rt)}{T} = 0$$

Because equation (2.3) has solution

$$\nu = \frac{1}{2\sqrt{T}\,(\sigma^2\tau + 1)} + \frac{1}{2}\sqrt{\left(-\frac{1}{\sqrt{T}\,(\sigma^2\tau + 1)}\right)^2 - 4.1.\frac{2\,(a_M - rt)}{T}}$$

or

$$\nu = \frac{1}{2\sqrt{T}\,(\sigma^2\tau + 1)} - \frac{1}{2}\sqrt{\left(-\frac{1}{\sqrt{T}\,(\sigma^2\tau + 1)}\right)^2 - 4.1.\frac{2\,(a_M - rt)}{T}}$$

so to determine estimation of parameter ν then the solutions in $x = \sigma^2$ also can be substituted to two solution above. Finally, the estimate of parameters ν and σ are held.

2.2. Moment Estimate

Analogous to Monte Carlo estimate, suppose that historical asset price data is available at equally spaced time values $t_i = i\,\Delta t$,so S_{t_i} is the asset price at time t_i. Defined $U_i = \log \frac{S_{t_i}}{S_{t_{i-1}}}$ and $\{U_i\}$ are independent. Parameters estimation of asset price volatility ν and the information flow rate σ of BS-BHM model using the method of moment as follows

Suppose that $t = t_n$ is the current time and that the M+1 is most current asset prices. $\{S_{t_{n-M}}, S_{t_{n-M+1}}, \ldots S_{t_{n-1}}, S_{t_n}\}$ is also available and by using the corresponding log rasio data which is $\{U_{n+1-i}\}_{i=1}^{M}$ then the first sample moment (m_1 = mean) and the second sample moment (m_2), see Higham, D.J.[5], Mutijah, Guritno, S., and Gunardi [8], Shao, J. [11], and Subanar [13] are

$$m_1 = \frac{1}{M}\sum_{i=1}^{M} U_{n+1-i} \tag{2.5}$$

and

$$m_2 = \frac{1}{M}\sum_{i=1}^{M} \left(U_{n+1-i}\right)^2 \tag{2.6}$$

Estimation of parameters in moment estimate can be obtained by making the k-th moment of the sample to be equal to the k-th moment of the model.

Suppose $\mu_1$dan μ_2 are the first moment and the second moment for BS-BHM model, then

$$\mu_1 = E(U_i) = rt - \frac{1}{2}\nu^2 T + \frac{1}{2}\frac{\nu\sqrt{T}}{\sigma^2\tau + 1} \tag{2.7}$$

and

$$\mu_2 = E\left(U_i^2\right) = Var(U_i) + \left(E(U_i)\right)^2$$
$$= \left(\frac{\sigma\tau\,\nu\sqrt{T}}{t(\sigma^2\tau + 1)}\right)^2\left(\sigma^2 t^2 + \frac{t(T-t)}{T}\right) - \left(rt - \frac{1}{2}\nu^2 T + \frac{1}{2}\frac{\nu\sqrt{T}}{\sigma^2\tau + 1}\right)^2 \tag{2.8}$$

From equation (2.5) and equation (2.7) are obtained equation belows

$$m_1 = rt - \frac{1}{2}\nu^2 T + \frac{1}{2}\frac{\nu\sqrt{T}}{\sigma^2\tau + 1}$$

or

$$\nu^2 - \frac{1}{\sqrt{T}\,(\sigma^2\tau + 1)}\,\nu + \frac{2\,(m_1 - rt)}{T} = 0 \tag{2.9}$$

and from equation (2.6) and equation (2.8) are also obtained equation

$$m_2 = \left(\frac{\sigma\tau\, v\sqrt{T}}{t(\sigma^2\tau+1)}\right)^2 \left(\sigma^2 t^2 + \frac{t(T\text{-}t)}{T}\right) - \left(rt - \frac{1}{2}v^2 T + \frac{1}{2}\frac{v\sqrt{T}}{\sigma^2\tau+1}\right)^2$$

$$= \left(\frac{\sigma\tau\sqrt{T}}{t(\sigma^2\tau+1)}\right)^2 \left(\sigma^2 t^2 + \frac{t(T\text{-}t)}{T}\right) v^2 - (m_1)^2$$

or

$$v^2 = \frac{m_2 + (m_1)^2}{\left(\frac{\sigma\tau\sqrt{T}}{t(\sigma^2\tau+1)}\right)^2\left(\sigma^2 t^2 + \frac{t(T\text{-}t)}{T}\right)} = \frac{m_2 + (m_1)^2}{\frac{\sigma^4\tau^2 T}{(\sigma^2\tau+1)^2} + \frac{\sigma^2\tau^2(T\text{-}t)}{t(\sigma^2\tau+1)^2}} = \frac{m_2 + (m_1)^2}{\frac{\sigma^4\tau^2 tT + \sigma^2\tau^2(T\text{-}t)}{t(\sigma^2\tau+1)^2}}$$

$$= \frac{t(\sigma^2\tau+1)^2(m_2 + (m_1)^2)}{\sigma^4\tau^2 tT + \sigma^2\tau^2(T\text{-}t)}$$

(2.10)

Substitution of equation (2.10) to equation (2.9) is obtained

$$\frac{m_2 + (m_1)^2}{\left(\frac{\sigma\tau\sqrt{T}}{t(\sigma^2\tau+1)}\right)^2\left(\sigma^2 t^2 + \frac{t(T\text{-}t)}{T}\right)} - \frac{1}{\sqrt{T}\,(\sigma^2\tau+1)} \cdot \frac{\sqrt{m_2 + (m_1)^2}}{\left(\frac{\sigma\tau\sqrt{T}}{t(\sigma^2\tau+1)}\right)\sqrt{\sigma^2 t^2 + \frac{t(T\text{-}t)}{T}}} + \frac{2(m_1 - rt)}{T} = 0$$

Making the same in denumerator to the equation is obtained

$$\frac{(m_2 + (m_1)^2)\, t^2(\sigma^2\tau+1)^2 - \sigma\tau t\sqrt{m_2 + (m_1)^2}\,\sqrt{\sigma^2 t^2 + \frac{t(T\text{-}t)}{T}} + 2(m_1 - rt)(\sigma\tau\sqrt{T})^2\left(\sigma^2 t^2 + \frac{t(T\text{-}t)}{T}\right)}{(\sigma\tau\sqrt{T})^2\left(\sigma^2 t^2 + \frac{t(T\text{-}t)}{T}\right)} = 0$$

By dividing to multiplying then $(m_2 + (m_1)^2)\, t^2(\sigma^2\tau + 1)^2 - \sigma\tau t$ $\sqrt{m_2 + (m_1)^2}\,\sqrt{\sigma^2 t^2 + \frac{t(T\text{-}t)}{T}} +$

$$2(m_1 - rt)(\sigma\tau\sqrt{T})^2\left(\sigma^2 t^2 + \frac{t(T\text{-}t)}{T}\right) = 0$$

Changing in form of equation the same as

$$\sigma\tau t\sqrt{m_2 + (m_1)^2}\,\sqrt{\sigma^2 t^2 + \frac{t(T\text{-}t)}{T}} = (m_2 + (m_1)^2)t^2(\sigma^2\tau + 1)^2 + 2(m_1 - rt)(\sigma\tau\sqrt{T})^2\left(\sigma^2 t^2 + \frac{t(T-t)}{T}\right)$$

and by multiplying to dividing is obtained

$$\sqrt{\sigma^2 t^2 + \frac{t(T\text{-}t)}{T}} = \frac{\sqrt{m_2 + (m_1)^2}\; t\,(\sigma^2\tau+1)^2}{\sigma\tau} + \frac{2(m_1 - rt)(\sigma\tau\sqrt{T})^2\left(\sigma^2 t^2 + \frac{t(T-t)}{T}\right)}{t\sqrt{m_2 + (m_1)^2}}$$

The left and right hands are squared

$$\sigma^2 t^2 + \frac{t(T\text{-}t)}{T} = \frac{m_2 + (m_1)^2\, t^2(\sigma^2\tau+1)^4}{\sigma^2\tau^2} + 4T(m_1 - rt)\left(\sigma^2 t^2 + \frac{t(T-t)}{T}\right)(\sigma^2\tau + 1)^2 + \frac{4(m_1 - rt)^2(\sigma^2\tau^2 T^2)\left(\sigma^2 t^2 + \frac{t(T-t)}{T}\right)^2}{t^2\,(m_2 + (m_1)^2)}$$

Thus, it is made in equation is to be equal to zero

$$\frac{(m_2 + (m_1)^2)\, t^2(\sigma^2\tau + 1)^4}{\sigma^2\tau^2} + 4T(m_1 - rt)\left(\sigma^2 t^2 + \frac{t(T-t)}{T}\right)(\sigma^2\tau + 1)^2 + \frac{4(m_1 - rt)^2(\sigma^2\tau^2 T^2)\left(\sigma^2 t^2 + \frac{t(T-t)}{T}\right)^2}{t^2\,(m_2 + (m_1)^2)} - \sigma^2 t^2 - \frac{t(T\text{-}t)}{T} = 0$$

Successively, by algebra tricks then it is held the final result is the polynomial equation as below

$$A\sigma^8 + B\sigma^6 + C\sigma^4 + D\sigma^2 + E = 0$$

where

$$A = (m_2 + (m_1)^2)\, t^2\tau^2 + 4T(m_1 - rt)\, t^2\,\tau^2 + \frac{4(m_1 - rt)^2\,\tau^2 T^2 t^2}{m_2 + (m_1)^2}$$

$$B = (m_2 + (m_1)^2)\, t^2\tau + 4T(m_1 - rt)\left(2t^2\tau + \frac{t\,(T-t)\,\tau^2}{T}\right) + \frac{8(m_1 - rt)^2\,\tau^2\, t\, T(T-t)}{m_2 + (m_1)^2}$$

$$C = 6\,(m_2 + (m_1)^2)\, t^2 + 4T(m_1 - rt)\left(t^2 + \frac{t\,(T-t)\,\tau}{T}\right) + \frac{4(m_1 - rt)^2\,\tau^2\,(T-t)^2}{m_2 + (m_1)^2}$$

$$D = \frac{4\left(m_2 + (m_1)^2\right) t^2}{\tau} + 4(m_1 - rt)t(T-t) - \frac{t\,(T-t)}{T}$$

$$E = \frac{\left(m_2 + (m_1)^2\right) t^2}{\tau^2}$$

Let $\sigma^2 = x$, then it is obtained the polynomial of four degree

$$Ax^4 + Bx^3 + Cx^2 + Dx + E = 0$$

The equation of polynomial of four degree has four solution in $x = \sigma^2$.

Further, to determine estimation of parameter ν then the solutions in $x = \sigma^2$ are substituted to equation (2.9) that is

$$\nu^2 - \frac{1}{\sqrt{T}\,(\sigma^2\tau + 1)}\nu + \frac{2\,(a_M - rt)}{T} = 0$$

Because equation (2.9) has solution

$$\nu = \frac{1}{2\sqrt{T}\,(\sigma^2\tau + 1)} + \frac{1}{2}\sqrt{\left(-\frac{1}{\sqrt{T}\,(\sigma^2\tau + 1)}\right)^2 - 4.1.\frac{2\,(a_M - rt)}{T}}$$

or

$$\nu = \frac{1}{2\sqrt{T}\,(\sigma^2\tau + 1)} - \frac{1}{2}\sqrt{\left(-\frac{1}{\sqrt{T}\,(\sigma^2\tau + 1)}\right)^2 - 4.1.\frac{2\,(a_M - rt)}{T}}$$

so to determine estimation of parameter ν then the solutions in $x = \sigma^2$ also can be substituted to the two solution above. Finally, estimation of parameters ν and σ are held too.

2.3. Application for Example of Real Data

Under BS-BHM and BS-BHM Updated model, estimation of historical volatility and the information flow rate of Microsoft (MSFT) shares for Monthly data are done by using Monte Carlo and moment estimates. For two parameters and two methods of estimates, it is assumed that the data corresponds to equally spaced points in time as in Higham, D.J [5], and Mutijah, Guritno, S., and Gunardi [8].

In Monte Carlo estimate, the monthly data runs over 5 years (T = 5 years) and has 60 asset prices (M = 59), so it has dt = T/M = 5/59 ≈ 0,084746. For the monthly data result in $a_M = -1{,}47 \times 10^{-3}$ and $b_M^2 = 1{,}056 \times 10^{-3}$. Under BS-BHM model, estimation based on Monte Carlo produces estimation of parameters ν and σ , they are

$\hat{\nu}_1 = 3.1822$, $\hat{\sigma}_1 = 3.9882$,
$\hat{\nu}_2 = -0.0796 + 0.6060i$, $\hat{\sigma}_2 = 1.0545 + 3.7081i$,
$\hat{\nu}_3 = -0.0796 - 0.6060i$, $\hat{\sigma}_3 = 1.0545 - 3.7081i$,
$\hat{\nu}_4 = 1.3451$, $\hat{\sigma}_4 = 0.1084$
$\hat{\nu}_5 = 2.1200$, $\hat{\sigma}_1 = 3.9882$,
$\hat{\nu}_6 = -0.1205 + 0.9014i$, $\hat{\sigma}_2 = -0.1205 + 0.9014i$,
$\hat{\nu}_7 = -0.1205 - 0.9014i$, $\hat{\sigma}_3 = 1.0545 - 3.7081i$,
$\hat{\nu}_8 = 0.8933$, $\hat{\sigma}_4 = 0.1084$

There are eight estimation of parameters ν and σ. They include four real number and four imaginary number. Therefore, estimation of parameters ν and σ are chosen four real number.

In moment estimate, the monthly data runs over 5 years (T = 5 years) and has 60 asset prices (M = 59), so it has dt = T/M = 5/59 ≈ 0,084746. For the monthly data result in $m_1 = -1{,}47 \times 10^{-3}$ and $m_2 = 1{,}1 \times 10^{-3}$. Under BS-BHM model, estimation based on the moment estimate produces estimation of parameters ν and σ, they are

$\hat{\nu}_1 = 3.1849$, $\hat{\sigma}_1 = 3.9911$,
$\hat{\nu}_2 = -0.0803 + 0.6057i$, $\hat{\sigma}_2 = 1.0537 + 3.7091i$,
$\hat{\nu}_3 = -0.0803 - 0.6057i$, $\hat{\sigma}_3 = -0.0803 - 0.6057i$,
$\hat{\nu}_4 = 1.3451$, $\hat{\sigma}_4 = 0.1085$
$\hat{\nu}_5 = 2.1218$, $\hat{\sigma}_1 = 3.9911$,
$\hat{\nu}_6 = -0.1215 + 0.9010i$, $\hat{\sigma}_2 = 1.0537 + 3.7091i$,
$\hat{\nu}_7 = -0.1215 - 0.9010i$, $\hat{\sigma}_3 = -0.0803 - 0.6057i$,
$\hat{\nu}_8 = 0.8933$, $\hat{\sigma}_4 = 0.1085$

The same as Monte Carlo estimate, there are eight estimation of parameters ν and σ too. They also include four real number and four imaginary number. Therefore, estimation of parameters ν and σ are chosen four real number too.

Whereas, to compare the results of estimation parameters under BS-BHM model with under BS-BHM Updated model, it can be seen in Mutijah, Guritno,S, and Gunardi [8].

3. Concluding Remarks

BHM model or BHM approach is modelled by Brody Hughston Macrina (BHM) under the assumption that market participants do not have acces to the information about the actual value of the relevant market variable. Brody Hughston Macrina defined an asset by its cash flow structure and then the associated market factor is the upcoming cash flow that is the upcoming dividend. Brody Hughston Macrina also built BHM model by a special condition which it is called Black Scholes model from an information-based perspective or it is called BS-BHM model in this paper. Further, the BS-BHM model that it is improved the result of Gaussian integral, especially in completing square is named BS-BHM Updated model. Both BS-BHM model and BS-BHM Updated model have lognormal distribution. BS-BHM model and BS-BHM Updated model also have the same as two parameter that is the volatility parameter ν and the information flow rate parameter σ. Estimation of the two parameters result an equation of polynomial of four degree under BS-BHM model and rerult a quadratic equation under BS-BHM Updated model. Application for real data of Microsoft shares results four value of estimation of parameters, while under BS-BHM Updated model result one value of estimation of parameter. All about BS-BHM Updated model can be seen Mutijah,Guritno, S, and Gunardi [7, 8].

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References

[1] Brody, D.C, Hughston, L.P, and Macrina, A.,2006,Information-Based Asset Pricing, *Working Paper*.

[2] Box-Tiao,1973, *Bayesian Inference in Statistical Analysis,Addison-Wesley Publising Company*,Department of Statistics University of Wisconsin.

[3] Caliskan, N.,2007,Asset Pricing Models:Stochastic Volatility and Information-Based Approaches,*Thesis*,Department of Financial Mathematics.

[4] Cochrane, J.H.,2000,*Asset Pricing*,Graduate Scholes of Business University of Chicago.

[5] Higham, D.J.,2004,*An Introduction to Financial Option Valuation Mathematics,Stochastics and Computation*, Cambridge University Press,Department of Mathematics University of Strathclyde.

[6] Macrina, A.,2006,An Information-Based Framework for Asset Pricing:X-Factor Theory and its Applications, *Dissertation*, King's College London.

[7] Mutijah, Guritno, S, and Gunardi,2012,A Black Scholes Model from an Information-Based Perspective by Brody Hughston Macrina,*International Conference on Statistics in Science, Business and Engineering*,Ready_ICSSBE2012-109

[8] Mutijah, Guritno,S, and Gunardi,2013,Estimation of Parameters on the BS-BHM Updated Model. *Journal Applied Mathematical Sciences*, Vol. 7, 2013, No. 72, 3555-3568.

[9] Rupert, D.,2004,*Statistics and Finance An Introduction,*Springer-Verlag New York,LLC.

[10] Shreve, S.E.,2004,*Stochastic Calculus for Finance II*.Springer Science+Business Media,Inc.

[11] Shao,J,2003,*Mathematical Statistics Second Edition*,Springer Science+Business Media,LLC,New York.

[12] Straub, W.O., 2009,A Brief Look at Gaussian Integrals,*Article*,Pasadena California.

[13] Subanar,2013,*Statistika Matematika,*Graha Ilmu,Yogyakarta.

EARLY DRUGS DETECTION TENDENCY FACTOR'S MODEL OF FRESH STUDENTS IN MATHEMATICS DEPARTMENT UI

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Abstract. In the last decade, Indonesia faced the increasing number of drug abuse very serious especially among young people. One of the people that can be attached to any drugs is a university student. The tendency of an attachment to drugs can be detected early. One of the tools that can be used to measure the tendency of an attachment to drugs is a psychological test. This tool can detect whether a person will be tied to drugs or not. The purpose of this paper is looking for a significant factors that affect this tendency and find the model based on those factors to determine whether a person will have a tendency to be tied on drug or not. The factors that are considered are gender, age, ethnic group, education of student's parent, employment status of student's parent, with whom respondents resided, score OAT (Obvious Attributes), score DEF (Defensiveness), score SAM (Supplemental Addiction Measure), and score Cor (Correctional). The methods that will be used here are clustering, decision tree, and discriminant analysis. The study is applied to fresh students in Mathematics Department, University of Indonesia, 2013.

Key words and Phrases : Narcotics, Clustering, Tree, Discriminant Analysis.

1. Introduction

A. Background

In the last decade, drugs continues to threaten the lives of the people in Indonesia. Based on data from Direktorat Tindak Pidana Narkoba Bareskrim Polri & Badan Narkotika Nasional (BNN), Maret 2012, in 2010 , there are 26.677 cases which is 17.897 include narcotics cases 1.181 include psikotropika cases, and 7.599 cases for other additive materials. In 2011, there is an increase of 11.69% which is there are 29.796 cases which is 19.128 include narcotics cases, 1.601 include psikotropika cases dan 9.067 cases for other additive materials.The increase continued to occur every year, where most of the victims of drugs abuse are students.

Drugs abuse provides a variety of negative impacts very seriously for your self, others, society, nation and the country. A drugs addict can spread HIV through free sex and also might experience death from an overdose. On the other hand, an addict often do criminal acts for get money to buy drugs, such as, stealing, robbing, even comitted murder. This caused drug abuse very dangerous.

Pollich stated that the most effective efforts to prevent drug abuse is early detection. He said that, " ... the majority of well-controlled studies suggest that early detection is associated with more positive and succesful treatment outcomes.". This detection can be used to identify a person's tendency to do drugs. This can lead to effects and more severe damage can be prevented immediately [4].

During this time, early detection is done with a urine toxicology test on every person suspected of involved drug abuse. However, a urine toxicology test was only able to detect any additive substances that last 48-72 hours in the human body, more than that of the urine test was not able to detect it again texicologi. This weakness can be anticipated with the use of a psychological test which this tool can distinguish someone who will be tends to be dependent on a drug or not [4].

Therefore, the factors that affect the tendency will be tied to the drug needs to be sought so that will be retrieved on a model for early detection of whether someone is likely to be tied to the drug or not. The factors that are considered are gender, age, ethnic group, education of student's parent, employment status of student's parent, with whom respondents resided, score OAT (Obvious Attributes), score DEF (Defensiveness), score SAM (Supplemental Addiction Measure), and score Cor (Correctional). Furthermore, the tendency of dependence on a drug called dependentnesss.

B. Research Problem

How to find a model for early detection of whether someone is likely to be tied to the drug or not as well as searching for factors that can affect the tendency will be tied to drugs?

C. Research Purpose

Looking for a significant factors that affect this tendency and find the model based on those factors to determine whether a person will have a tendency to be tied on drug or not.

D. Research Methods

a. Respondents : fresh students in Mathematics Department, University of Indonesia, 2013 (108 students)
b. Measuring instruments is the Likert scale with the measurement tool to detect whether someone is likely to be tied to a drug or not. Realibility of these tool is 0.839 and all items are valid items.
c. The technique of data collection by spreading a questionnaire.
d. The sample technique used is purposive sampling.
e. A method of analysis data are clustering, tree, and diskriminant analysis.
f. Operational Definition of The Variables

Some important terms used in this paper are:

1. Dependentness
   The level of person's who is dependence in drugs. High scores on this score indicates a higher tendency to have himself against drug abuse.
2. OAT (Obvious Attributes)
   OAT scale scores reflect an individual's tendency to indorse statements of personal limitation. High score may be relatively able to recognize in themselves what are sometimes terms "character defects", and they tend to endorses statements suggesting personal limitation.
3. SAM (Supplemental Addiction Measure)
   This scale can measure the level of dependency to something other than drugs.
4. Cor (Correctional)
   The COR scale can be used to assess the client's level of risk for legal problems. Clients who have elevated COR scale scores show response patterns similar to adolescents who have been refered to correctional program.

2. Main Results

a. Descriptive Statistics

- Gender : Male (39 students), female (69 students)

Table 1

Gender

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Male	39	36.1	36.1	36.1
	Female	69	63.9	63.9	100.0
	Total	108	100.0	100.0	

- Ethnic Group : Jawa, Sunda, Betawi (59 students), others (32 students)

Table 2

Ethnic Group

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Jawa/Sunda/Betawi	59	54.6	64.8	64.8
	Others	32	29.6	35.2	100.0
	Total	91	84.3	100.0	
Missing	System	17	15.7		
Total		108	100.0		

- Education of student's parents : < = high school (37 students), D3 (3 students), S1 (50 students), > = S2 (11 students), others (4 students), missing (3 students)

Table 3

Education of Student's Parents

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	<=high school	37	34.3	35.2	35.2
	D3	3	2.8	2.9	38.1
	S1	50	46.3	47.6	85.7
	>=S2	11	10.2	10.5	96.2
	Others	4	3.7	3.8	100.0
	Total	105	97.2	100.0	
Missing	System	3	2.8		
Total		108	100.0		

- Employment's status of students's parents : no job (3 students), goverment employee (23 students), private employee (41 students), enterpreuner (35 students), others (6 students)

Table 4

Employment's status of Student's Parents

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	No job	3	2.8	2.8	2.8
	Goverment Employee	23	21.3	21.3	24.1
	Private Employee	41	38.0	38.0	62.0
	Entrepreneur	35	32.4	32.4	94.4
	Others	6	4.6	4.6	100.0
	Total	108	100.0	100.0	

- With whom respondents resided : parents / family (51 students), homestay (44 students), dorm (13 students)

Table 5

With whom Respondents Resided

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Parents/family	51	47.2	47.2	47.2
	Homestay	44	40.7	40.7	88.0
	Dorm	13	12.0	12.0	100.0
	Total	108	100.0	100.0	

- Marital's status of students's parents: married (97 students), divorced dead (4 students), divorced life (7 students)

Table 6

Marital's Status of Students's Parents

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Married	97	89.8	89.8	89.8
	Divorced Dead	4	3.7	3.7	93.5
	Divorced Life	7	6.5	6.5	100.0
	Total	108	100.0	100.0	

b. Drugs dependence based on dependentness
By using two step clustering method, the author will classify the respondents based on variable dependentness. It turns out, with Two Step Clustering obtained 2 categories of students who will have a low dependency on drugs and high dependency of drugs, namely :
   1. Low (49 students), score 23-31, mean score 29.37 (valid item=16)
   2. High (56 students), score 32-49, mean score 35.9643 (valid item=16)

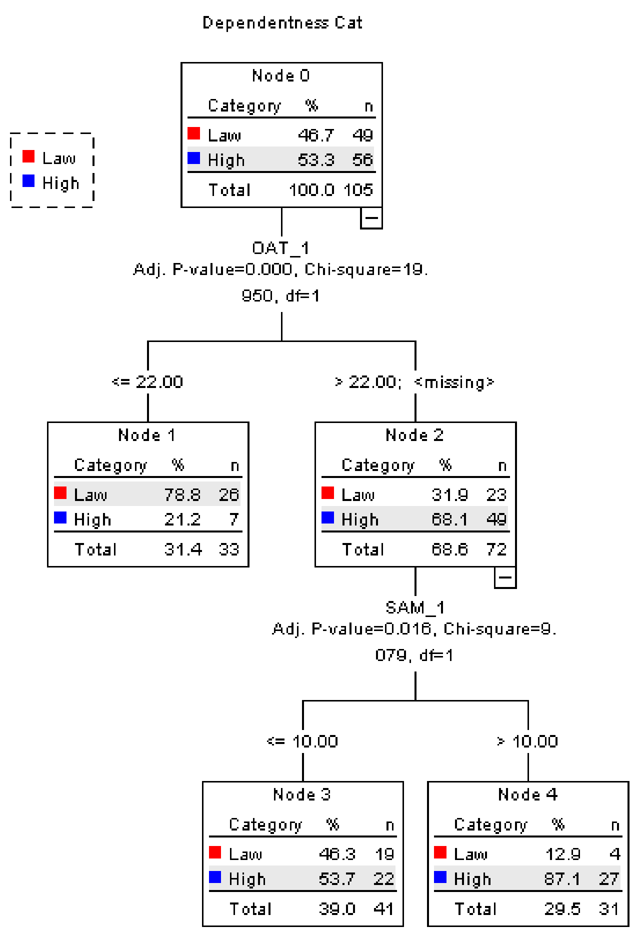
c. Factors that distinguish dependentness variable
The factors that distinguish the variable dependentness is sought by using a tree diagram which the result obtained in Figure 1. So from the tree can be conclude that factors that distinguishes the dependentness is OAT and SAM. The main factor that most distinguishes is OAT which is outlined as follows :
   - OAT $\leq$ 22 (low OAT), valid item =10
     There are 26 students (78.8%) which have a low dependency of drugs and there are 7 students (21.2%) which have a high dependency of drugs.
   - OAT > 22 (high OAT), valid item = 10
     There are 23 students (31,9%) which have a low dependency of drugs and there are 49 students (68.1%) which have a high dependency of drugs.

Other factors besides OAT is SAM which is outlined as follows :

- SAM ≤ 10 (low SAM), valid item = 5
  There are 19 students (46.3%) which have a low dependency of drugs and there are 22 students (53.7%) which have a high dependency of drugs.
- SAM > 10 (high SAM), valid item =5
  There are 4 students (12.9%) which have a low dependency of drugs and there are 27 students (87.1%) which have a high dependency of drugs.

d. Group profile characteristic students based on variables OAT and SAM
   To classify students who will have low and high dependency on drugs, it can be seen with variable OAT and SAM. The result analysis can elaborated as follows :
   1. If a students who have score OAT > 22 and score SAM ≤ 10 then there are 19 students (46.3%) have low dependentness and 22 students (53.7%) have high dependentness.
   2. If a students have score OAT > 22 and score SAM > 10 then there are 4 students (12.9%) have low dependentness and 27 students (87.1%) have high dependentness

Figure 1. Tree



e. Discriminant

Using discriminant analysis, the authors make a function that distinguishes student tendency on drugs. The result analysis can elaborated as follows :

1. Tests of Equality of Group Means
   - OAT : Wilks' Lambda = 0.854, F = 17.274, Sig. = 0.000 < 0.15, conclusion : variabel OAT significant
   - SAM : Wilks' Lambda = 0.791, F = 26.631, Sig. = 0.000 < 0.15, conclusion : variabel SAM significant
2. Box's M test of equality of covariance matrics : Sig. = 0.157 > 0.15 so assuming a covariance matrix are fulfilled.
3. Wilk's Lambda : Sig. = 0.000 < 0.15 so it can be concluded that the discriminant functions distinguishes the group dependentness.

So, the unstandardized estimate of discriminant function is :

$$D = -5.553 + 0.087\ OAT + 0.344\ SAM \quad (1)$$

And Functions of Group Centroid is

1. Low = - 0.556
2. High = 0.504

Discriminat function can be used to determine whether one new students has a high or low dependentness of drugs.

Cutoff value = $(N_{high}.\ Centroid_{low} + N_{low}.\ Centroid_{high})/\ N_{high} + N_{low}$

From data, it is found tahat cutoff value = -0.06.

When new student has D score > -0.06, it can be concluded that the students will has high score depentdentness to drugs.

3. Concluding Remarks

Tendency of dependence on the drugs can be detected early by using a measuring instrument psychology. The result of test states that there are 56 fresh students in Mathematics Departement,University of Indonesia, 2013 who will have high score dependentness. The factors that distinguishes dependentness on drugs are OAT and SAM. Students who have high score of dependentness to drgus are OAT > 2.2/item & SAM > 2,2/item (87.1%) and OAT > 2.2/item & SAM <= 2,2/item (53.7%). Suggestions for Departement Mathematics UI is it has to concern about this results.

References

[1] Quinlan, J.R., 1986, *Induction of Decision Trees*, Kluwer Academic Publishers, Boston.

[2] Rencher, A.C., 2002, *Methods of Multivariate Analysis*, John Wiley & Sons.

[3] Sharma, Subhash, 1996, *Applied Multivariat Techniques*, John Wiley & Sons.

[4] Vincent, Stephen, 2003, Adaptasi Subtances Abuse Subtle Screening Inventory (SASSI-A2) : Sebagai Alat Bantu Deteksi Dini Level Penyalahgunaan Narkoba Anak Remaja (thesis)

COMPARISON OF LOGIT MODEL AND PROBIT MODEL ON MULTIVARIATE BINARY RESPONSE

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Abstract. On univariate binary response, Logit model is better interpretation compared to Probit model. Logit model and Probit model may be used to analyze same data sets for the same purpose but which model can perform better analysis on multivariate binary data is an interesting topics to be studied. In this study, a comparison of multivariate binary probit and logit models via a simulation study was performed under different correlations between dependent variables. We assume that each of n individual observed T response. Y_{it} is t^{nd} response on i^{nd} individual/subject and each response is binary. Each subject has covariate X_i (individual characteristic) and covariate Z_{ijt} (characteristic of alternative j). Individual response i can be represented by $\mathbf{Y}_i=(Y_{i1},....,Y_{iT})$, Y_{it} is t^{nd} response on i^{nd} individual/subject and each response is binary. In order to simplify, we choose one of individual characteristics and alternative characteristics. We studied effects of correlations using data simulation. General Estimating Equations (GEE) was used to estimate the parameters in this study. Data were generated by using software R.2.8.1 as well as the estimation on the parameters. Based on the result, it can be concluded that estimator in the logit model is equivalent to 1.63 on the probit model. Estimator of the correlation base on Chaganty-Joe is more accurate compared to GEE base on Liang-Zeger.

Key words and Phrases : Random Utility Model, GEE, Simulated maximum likelihood estimator, Newton-Raphson, GHK.

1. Introduction

Investigators often encounter a situation in which plausible statistical models for observed data require an assumption of correlation between successive measurements on the same subjects (longitudinal data) or related subjects (clustered data) enrolled in clinical studies. Statistical models that fail to account for correlation between repeated measures are likely to produce invalid inferences since parameter estimates may not be consistent and standard error estimates may be wrong[1]. Statistical methods that appropriate for analyzing repeated measures include generalized estimating equations (GEE) and multi-level/mixed-linear models[2]. GEE involves specifying a marginal mean model relating the response to the covariates and a plausible correlation structure between responses at different time periods (or within each cluster). Estimated Parameter thus obtained are consistent irrespective of the underlying *true* correlation structure, but may be inefficient when the correlation structure is misspecified[2]. GEE parameter estimates are also sensitive to outliers[2,3]. Summary statistics derived from the likelihood ratio test can be used to check model adequacy in cross-sectional data analyses[1,4,5]. For mixed linear models, the process is often not straightforward due to the complexities involved[6]. Model selection is difficult in GEE due to lack of an absolute goodness-of-fit test to help in choosing the "best" model among several plausible models [4,5,7]. For repeated binary responses, Barnhart and Williamson[5] and Horton *et al.*[4] proposed ad-hoc goodness-of-fit statistics which are extensions of the Hosmer and Lemeshow method for cross-sectional logistic regression models[4,5,8].

Frequently, some dependent variables are observed in each individual. This observation results the multivariate data. Research of multivariate binary response models still gets a little attention, however the applications of multivariate binary response model are mostly extensive. GEE can be implemented on multivariate binary response. Variable Y_{it} with i=1,..n and t=1,..T in panel data (longitudinal) refer to the same variable. In multivariate binary response, Y_{it} refer to the different variable (T variables)[2]. Nugraha *et al.* [9,10] has tested properties of estimator of bivariate logistic regression using MLE and GEE. Both of MLE and GEE are consistent. Logistic model on multivariate binary data using GEE are more efficient compared to the univariate approximation. From simulation data, it was concluded that GEE was better than MLE, specifically GEE able to accommodate the correlations and the GEE's estimator was more precise than MLE.

Furthermore, Nugraha *et al*.[11,12] discussed estimating parameter of Probit model on multivariate binary response using simulated maximum likelihood estimator (SMLE) methods to estimate the parameter based on Geweke-Hajivassiliou-Keane (GHK) simulator. From computational side, simulation method applicable for Probit model is need to be developed to overcome the limitation of GHK method. For this limitation, Nugraha[13] have proposed mixed logit model. From simulation data, he conclude that mixed logit model is better than logit model.

Based on the fact that GEE was acceptable than MLE and is widely available in many statistical software applications, in this study, we compare the probit model and logit model on multivariate binary response using simulation data. In the R.2.8.1 program, the logit and probit models can be obtained by using library(*geepack*) and library(*mprobit*). The *geepack* is the GEE that is based on Liang-Zeger and the *mprobit* is the GEE that is based on Chaganty-Joe. Generating data and estimating paremater using R 2.8.1 software[14].

2. Utility Model

We assume that each of n individual is observed for T response in that Y_{it} is t^{nd} response on i^{nd} individual/subject and each response is binary. Response for i^{nd} individual can be represented by following statement:

$$Y_i = (Y_{i1},....,Y_{iT})$$

that is a vector of 1xT. $Y_{it} = 1$ if i[nd] subject and t[nd] response choose the first alternative and $Y_{it} = 0$ if the subject choose the second alternative. Each subject has covariate of X_i (individual characteristic) and covariate of Z_{ijt} (characteristic of alternative j=0,1). In order to simplify, one of individual characteristic and one of characteristic of alternative were chosen.

Utility of subject of i choose alternative of j on response t is

$$U_{ijt} = V_{ijt} + \varepsilon_{ijt} \quad \text{for } t=1,2,...,T \text{ ; } i=1,2,...,n \text{ ; } j=0,1 \tag{1}$$

where

$$V_{ijt} = \alpha_{jt} + \beta_{jt}X_i + \gamma_t Z_{ijt}$$

U_{ijt} is utility that it is latent variable and V_{ijt} is named representative utility. In Random Utility Model (RUM), assumption that decision maker (subject) choosing alternative based on maximize utility, so equation (1) can represented in different of utility,

$$U_{it} = V_{it} + \varepsilon_{it} \tag{2}$$

where $V_{it} = (\alpha_{1t}-\alpha_{0t}) + (\beta_{1t}-\beta_{0t})X_i + \gamma_t(Z_{i1t} - Z_{i0t})$ and $\varepsilon_{it} = \varepsilon_{i1t} - \varepsilon_{i0t}$.

Association between Y_{it} and U_{it} is

$$y_{it} = 1 \Leftrightarrow U_{it} > 0 \Leftrightarrow -V_{it} < \varepsilon_{it} \text{ and } y_{it} = 0 \Leftrightarrow U_{it} < 0 \Leftrightarrow -V_{it} > \varepsilon_{it}.$$

Probability of subject i choose ($y_{i1} = 1,...., y_{iT} = 1$) is

$$= P(0 < U_{i1},..., 0 < U_{iT}) = P(-V_{i1} < \varepsilon_{i1},..., -V_{iT} < \varepsilon_{iT})$$

$$P(y_{i1} = 1,...., y_{iT} = 1) = \int_{\varepsilon i} I(-V_{it} < \varepsilon_{it}).f(\varepsilon_i)d\varepsilon_i \qquad \forall t \tag{3}$$

where $\varepsilon'_i = (\varepsilon_{i1},..., \varepsilon_{iT})$. The value of probability is multiple integral T and depend on parameters α, β, γ distribution ε.

Logit model derived by asumption that ε_{ijt} have extreme value distribution and independence each other. Density of extreme value (Gumbel) is

$$f(\varepsilon_{iit}) = \varepsilon^{-\varepsilon_{ijt}} \varepsilon^{-\varepsilon^{-\varepsilon_{ijt}}} \tag{4}$$

Marginal probability (for some t and i) is

$$P(y_{it} = 1) = \pi_{it} = \frac{\exp(V_{i1t})}{[\exp(V_{i0t}) + \exp(V_{i1t})]} \tag{5}$$

Probit model derived by assuming that vector $\varepsilon_i^{'}$ has a multivariate normal distribution with the mean of null and covarians of Σ. Density of ε_i is

$$f(\varepsilon_i) = \phi(\varepsilon_i) = \frac{1}{(2\pi)^{T/2} \mid \Sigma \mid^{1/2}} \exp[-\frac{1}{2}\varepsilon_i^{'}\Sigma^{-1}\varepsilon_i \ ] \tag{6}$$

Marginal probability (for some t and i) is

$$\pi_{it} = P(y_{it}=1|X_i,Z_i) = P(-V_{it} < \varepsilon_{it}) = \Phi(V_{it}) \tag{7}$$

where $\Phi(V_{it}) = \int_{-\infty}^{V_{it}} \frac{1}{(2\pi\sigma_t^2)^{1/2}} \exp[-\frac{1}{2\sigma_t^2}\varepsilon_{it}^2] d\varepsilon_{it}$

3. Overview of GEE

Marginal models are often fitted using the GEE methodology, whereby the relationship between the response and covariates is modeled separately from the correlation between repeated measurements on the same individual[2]. The correlation between successive measurements is modeled explicitly by assuming a "correlation structure" or "working correlation matrix". The assumption of a correlation structure facilitates the estimation of model parameters[2]. Examples of working correlation matrices include: exchangeable, auto-regressive of order 1 (AR(1)), unstructured, and independent correlation structures[2]. For binary data, correlation is often measured in terms of odds ratios[15]. Details of the correlation structure and response-covariate relationship are included in an expression known as the *quasi-likelihood function*[2], which is iteratively solved to obtain parameter estimates. Estimates obtained from the *quasi-likelihood function* are efficient when the true correlation matrix is closely approximated. In other words, the large-sample variance of the estimator reaches a Cramer-Rao type lower bound[3].

GEE for θ can present in form

$$G(\theta) = \sum_{i=1}^{n} W_i \Delta_i S_i^{-1} (Y'_i - \pi'_i) = 0 \tag{8}$$

$W_i = diag\left(\begin{pmatrix} 1 \\ X_i \\ (Z_{i11} - Z_{i01}) \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ X_i \\ (Z_{i1T} - Z_{i0T}) \end{pmatrix}\right)$ and

$\Delta_i = diag\left(\pi_{i1}(1-\pi_{i1}) \quad \ldots \quad \pi_{iT}(1-\pi_{iT})\right)$

$\mathbf{Y}_i = (Y_{i1},\ldots,Y_{iT})$; $\boldsymbol{\pi}_i = (\pi_{i1},\ldots,\pi_{iT})$; $\mathbf{S}_i = \mathbf{A}_i^{1/2}\mathbf{R}_i\mathbf{A}_i^{1/2}$;

$\mathbf{A}_i^{1/2} = diag\left(\sqrt{Var(Y_{i1})} \quad \ldots \quad \sqrt{Var(Y_{iT})}\right)$

$\mathbf{R}_i$ is working correlation matrix $\mathbf{Y}_i$ and $\mathbf{W}_i$ is an observation matrix.

To estimate R_i, Liang and Zeger[16] use vector of empirical corelation r_i with

$$r_{ist} = \frac{(Y_{is} - \pi_{is})(Y_{it} - \pi_{it})}{[\pi_{is}(1-\pi_{is})\pi_{it}(1-\pi_{it})]^{1/2}} \tag{9}$$

r_{ist} is usbias estimator for ρ_{ist} with i = 1,2,..n and s,r=1,2,…T.

In probit model, Chaganti and Joe[17] use

$$Kor(Y_{is}, Y_{it}) = \frac{\Phi(V_{is}, V_{it}; \rho_{st}) - \Phi(V_{is})\Phi(V_{it}))}{[\Phi(V_{is})(1-\Phi(V_{is}))\Phi(V_{it})(1-\Phi(V_{it}))]^{1/2}} \tag{10}$$

to estimate R_i. If $\rho_{ist}= \rho_{st}$ for all i then

$$\hat{\rho}_{st} = \frac{1}{n}\sum_{i=1}^{n} r_{ist} \tag{11}$$

Equation (8) and (11) can be solved simultaneously for θ and ρ.

4. Generating Simulation Data

We will generate simulation data with T=3. Then, the equations of utility are

$$U_{i0t} = \alpha_{0t} + \beta_{0t}X_i + \gamma_t Z_{i0t} + \varepsilon_{i01} \text{ and } U_{i1t} = \alpha_{1t} + \beta_{1t}X_i + \gamma_t Z_{i1t} + \varepsilon_{i11} \tag{12}$$

for i=1,...,N; j=0,1 and t=1,...,3; ε_{ijt} ~Extreme Value Type I for logit model and ε_{ijt}~N(0,1) for probit model. Equation (12) can be presented in difference of utility $U_{it} = U_{i1t}–U_{i0t}$. On logit model, equations of utility difference are

$$U_{it} = \alpha_t + \beta_t X_i + \gamma_t Z_{it} + \varepsilon_{it} \tag{13}$$

where $Z_{it} = (Z_{i1t} – Z_{i0t})$; $\alpha_t= \alpha_{0t}- \alpha_{1t}$;$\beta_t =\beta_{0t} - \beta_{1t}$.

We generate data on α_1=-1, α_2=1, α_3=-1; β_1 = 0.5, β_1 = -0.5, β_t = 0.5, γ_1=0.3, γ_2=-0.3, γ_3=0.3 and some of correlations ρ =0;0.1;0.2;...; 0.9 using program R.2.8.1. Utility 1 (U_{i1}) was correlated with utility 2 (U_{i2}) and both utility is not correlated with utility 3 (U_{i3}). Data 1 are obtained from ε_{it}~extreme value and Data 2 are obtained from ε_{it}~N(0,1). For each of the data simulation, we estimate parameter using logit model and probit model. GEE-1 are estimator of logit model based on Liang-Zeger. GEE-2 are estimator of probit model based on Liang-Zeger. GEE-3 are estimator of probit model based on Chaganty-Joe.

Those data can be further analyzed by using the program *geepack* and *mprobit*, so the utility must be transformated in the form of:

$$U_i = \sum_{t=1}^{T}\left(\alpha_t D_{it} + \beta_t X_i D_{it} + \gamma_t (Z_{i1t} - Z_{i1t})D_{it}\right) \tag{14}$$

where D_{ir} is *dummy* variable. D_{ir} =1 for r=t and D_{ir} =0 for r≠t, r=1,2,3. So

If t=1 then $D_{i1} = 1$ and $D_{i2} = D_{i3} = 0$. If t=2 then $D_{i2} = 1$ and $D_{i1} = D_{i3} = 0$. If t=3 then $D_{i3} = 1$ and $D_{i1} = D_{i2} = 0$

$$U_i = \alpha_t + \beta_t X_i + \gamma_t(Z_{i1t}- Z_{i0t})..$$

5. Main Result

Discrete Choice Model (DCM) was prepared based on the of the error distribution. So far there is no method to test the assumption of truth because the utilities(U_i) is also a latent variable can not be known by researchers in value.

5.1. Efek of ε variation to the Estimator

Logit model is constructed based on the assumption that variance ε_{it} is valued with $\pi^2/3$. The value of ε_{it} variance to the estimator is presented bellow. Based on the simulation data, it can be remarked that the value of variance ε_{it} give influence to the estimator. The bigger deviation of variace ε_{it} (from $\pi^2/3$) can affect to the resulted more bias estimator (bigger deviation). Suppose that $Var(\varepsilon_{it}) = \sigma^2$, where the utility model is

$$U_{it} = V_{it} + \varepsilon_{it}$$

Logit model use the assumption that value of error variance is $\pi^2/3$, so the utility model that will be estimated is

$$\frac{\pi}{\sigma\sqrt{3}}U_{it} = \frac{\pi}{\sigma\sqrt{3}}V_{it} + \frac{\pi}{\sigma\sqrt{3}}\varepsilon_{it}$$

$$\widetilde{U}_{it} = \widetilde{V}_{it} + \widetilde{\varepsilon}_{it} \text{ where } \frac{\pi}{\sigma\sqrt{3}}Var(\widetilde{\varepsilon}_{it}) = \frac{\pi^2}{3}$$

So, the estimator resulted will be deviated for $(1-\frac{\pi}{\sigma\sqrt{3}})\beta$.

5.2. Effect of Correlation to the Estimator

In advance simulation, data were generated on n=1000 with 5 replications on each correlations of utility (0 to 0.9). From the simulation (Figure 1 to Figure 10), it can be concluded that :

a. Estimator GEE-1 (in the logit model) is equivalent to 1.63 GEE-2 on the probit model. From Data 2, On the Probit model using the assumption that error (ε_{it}) having normal standard distribution, the value of correlation between utilities give no effect to the estimator properties. Estimator in the logit model is equivalent to 1.64 on the probit model. This is caused by differences in the size of the variansi error on logit model ($\sigma^2=\pi^2/6\cong1.645$) and variansi error in probit models ($\sigma^2=1$) (Table 1, Table 2 and Table 3).
b. Estimator of the correlation using GEE-3 is the most accurate (small bias) compared to GEE-1 and GEE-2.. The estimator is not affected by the faulty of error distribution assumptions. On GEE-1 and GEE-2, the estimator of correlation tends to underestimate.
c. Estimator of the parameter α using GEE-3 is not accurate compared to GEE-1 and GEE-2. But, estimator of the parameter β and γ using the third method the results are relatively the same.
d. Utility 1 (U_{i1}) was correlated with utility 2 (U_{i2}) and both utility is not correlated with utility 3 (U_{i3}). Therefore the value of correlation was only give effect to the parameters whitin U_{i1} and U_{i2}. On both utilities, parameter estimation on the model have big deviation comparable to the correlation value. Therefore the value of correlation was only give effect to

the parameters whitin U_{i1} and U_{i2} on Logit and GEE-1. Estimator of the parameter β and γ using GEE-2 and GEE-3 are more accurate than Logit and GEE-1

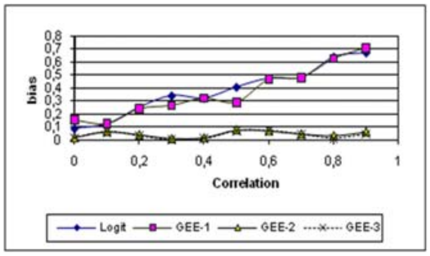

Figure 1. Bias of α_1

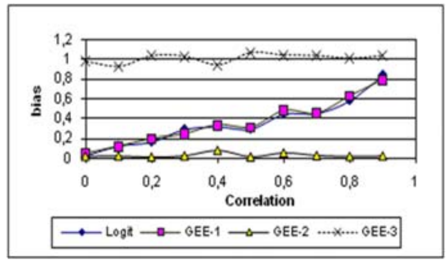

Figure 2. Bias of α_2

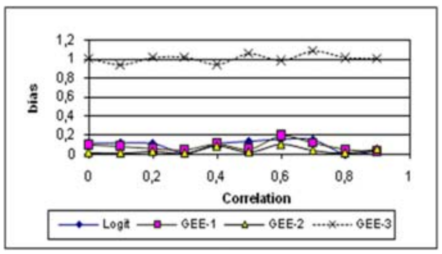

Figure 3. Bias of α_3

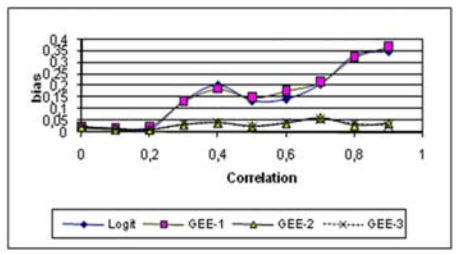

Figure 4. Bias of β_1

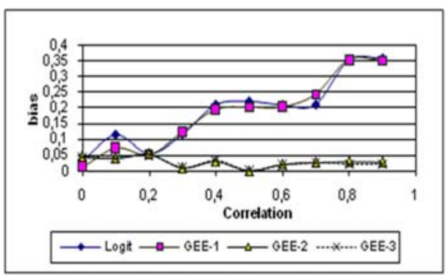

Figure 5. Bias of β_2

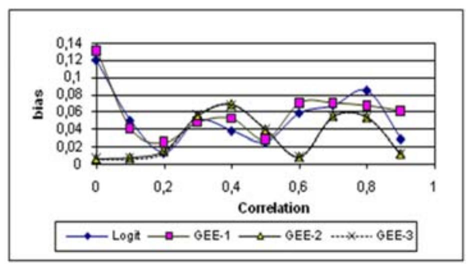

Figure 6. Bias of β_3

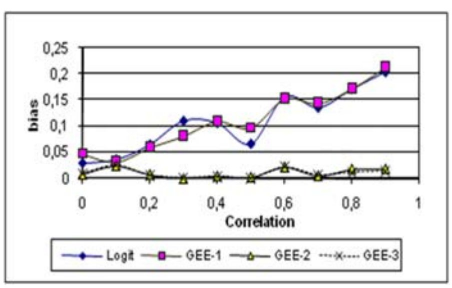

Figure 7. Bias of γ_1

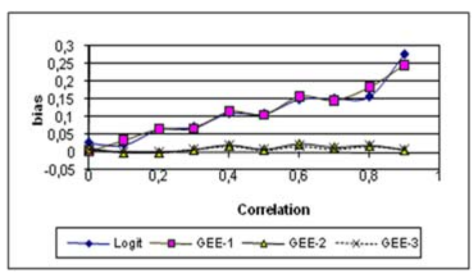

Figure 8. Bias of γ_2

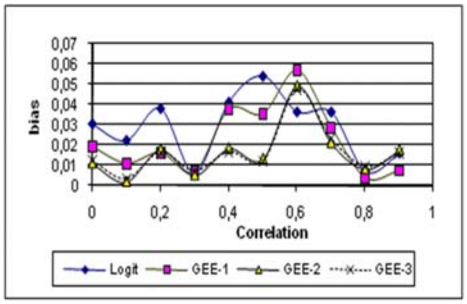

Figure 9. Bias of γ_3

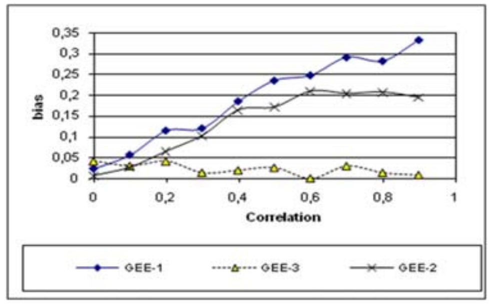

Figure 10. Bias of ρ

6. Concluding Remarks

Based on the results, it can be concluded that estimator in the logit model is equivalent to 1.63 on the probit model. Estimator of the correlation base on Chaganty-Joe is more accurate compared to GEE base on Liang-Zeger. So, we recommended to estimate correlation using GEE base on Chaganty-Joe and then using GEE base on Liang-Zeger to estimate coefisien regression.

References

[1] Dobson A, 2002, *An Introduction to Generalized Linear Models* Florida: Chapman & Hall/CRC.

[2] Diggle PJ, Heagerty P, Liang K, Zeger SL, 2002, A*nalysis of Longitudinal Data,* Second edition. Oxford: Oxford University Press.

[3] Qu, A., Lindsay, B. G. & Li, B., 2000, Improving generalised estimating equations using quadratic inference functions. Biometrika, 87, 823-836.

[4] Horton NJ, Bebchuk JD, Jones CL, Lipsitz SR, Catalano PJ, Zahner GE, Fitzmaurice GM., 1999, Goodness of fit for GEE: An example with mental health service utilization, *Stat Med*, 18(2):213-222.

[5] Barnhart H. X.,; Williamson J. M.., 1998, Goodness-of-fit tests for GEE modeling with binary responses. Biometrics 54(2):720–729.

[6] Schabenberger O., 2004, *Proceedings of the twenty-Ninth Annual SAS Users Group International Conference: May 9–12, 2004; Montreal* Cary, NC: SAS Institute Inc;189-29.

[7] Heagerty PJ, Zeger SL. 2000, Marginalized multilevel models and likelihood inference. Stat Sci. ;15: 1–26.

[8] Hosmer DW, Lemeshow S., 1980, Goodness-of-fit tests for the multiple logistic regression model. Communications in Statistics. A9:1043–1069.

[9] Nugraha J., Haryatmi S., Guritno S, 2009, Estimating Parameter of Logit Model on Multivariate Binary Response Using MLE and GEE. *Jurnal Ilmu Dasar*, Vol. 10. No. 1 , FMIPA Universitas Jember.

[10] Nugraha J., Guritno S., Haryatmi S. 2009, A Comparison of MLE and GEE on Modeling Binary Panel Response, ICoMS 3th IPB, 2008.

[11] Nugraha J., Guritno S., Haryatmi S., 2010, Probit Model on Multivariate Binary Respons Using SMLE, *Jurnal Ilmu Dasar*, FMIPA Univ. Jember,.

[12] Nugraha J., 2011, Simulation Study Of MLE on Multivariate Probit Models, Proceedings of The 6th SEAMS-UGM Conference 2011, FMIPA UGM.

[13] Nugraha J., 2011, Mixed Logit Model on Multivariate Binary Response Using Maximum Likelihood Estimator and Generalized Estimating Equations. Asian Journal of Mathematics and Statistics. Vol 4 (3) : 109-123.

[14] R Development Core Team, 2011, R : A Language and Enviroment Statistical Computing. R Foundation for Statistical Computing, Vienna, Austria.

[15] Fitzmaurice G.M., Laird N.M., Ratnitzky, A.G.,1993, Regression Models for Discrete Longitudinal Responses. *Statistical Science* Vol. 8 No. 3. 284 – 309.

[16] Liang K.Y., and Zeger S.L.,1986, Longitudinal Data Analysis Using Generalised Linear Models, *Biometrika 73, 13-22.*

[17] Chaganty NR. and Joe H.,2004, Efficiency of Generalized Estimating Equations for Binary Responses, *Journal of the Royal Statistical Society. Series B (Statistical Methodology)*, Vol. 66, No. 4, pp. 851-860.

MULTISTATE HIDDEN MARKOV MODEL FOR HEALTH INSURANCE PREMIUM CALCULATION

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Abstract. Health conditions over time can be modelled using multistate Markov model. However, the information about health conditions is not always available, but there is another information related to this condition. This paper presents hidden Markov model to estimate transition intensity and observation probability for multistate model where the true state is not observed. The estimation of transition intensity and transition probability will be used to calculate health insurance premium. By using this method, it is expected to get an appropriate premium value. This method will be applied to patient's visit in a clinic in West Java.

Key words and Phrases : Multistate model, hidden Markov model, health insurance premium.

1. Introduction

A multi-state model is a model for a stochastic process which occupies one of a set of discrete states at any time. The inference in multi-state models is traditionally performed under a Markov assumption for which past and future are independent given its present, Meira-Machado [9]. Sometimes, states of the process are not observed, but there there is a finite set of signals, and that a signal from the set is emitted each time the Markov chain enters a state. A model in which the sequence of signals is observed, while the sequence of underlying Markov chain states is unobserved, is called a hidden Markov model, Ross [11].

Hidden Markov model is commonly used in areas such as speech and signal processing, Juang and Rabiner [8], and the analysis of biological sequence data, Durbin *et al.* [4]. In health, Chen *et al.* [3], described a hidden Markov model for breast cancer screening, Satten and Longini [12] used hidden Markov models for the progression between stages of HIV infection, and Jackson *et al*. [5] used hidden Markov models in the study of abdominal aortic aneurysms by ultrasonography.

In health insurance, the premium value can be calculated based on the multistate model. In the process of calculation required data of the patients' health status over time. However, this data is not always available but it is possible that there are other variables that can describe the health status of the patients. Therefore, in this study utilized the results of estimation of hidden Markov models to calculate health insurance premiums.

2. Multistate Model

Individual's health condition from time to time subject to change. Changes in these conditions generate a data consists of the time of the event and the type of event that occurred. A model, called the multistate model, is suitable for this kind of data. For example, a simple multistate model that consists of 3 state, "1: healthy", "2: diseased", and "3: dead", is illustrated in Figure 1.

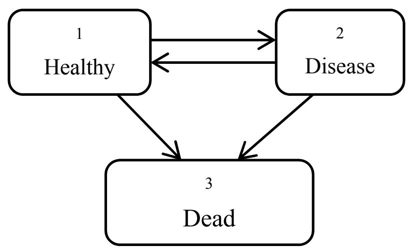

Figure 1 The three-state model

In the model ilustrated above, a healthy person is possible to make a transition into diseased or dead. A diseased person is possible to make a transition into dead, but a dead person is impossible to life again, so dead state is called terminal event or absorbing state, where once entered they are never left.

In general multistate model, the state of the individual at time t is denoted by $X(t)$, $t \geq 0$. Transition rate from state i to state j at time t is indicated by the transition intensity $q_{ij}(t, F_t)$,

$$q_{ij}(t, F_t) = \lim_{h \to 0} \frac{P\left(X(t+h) = j \middle| X(t) = i, F_t\right)}{h}$$

where F_t is the observation history of the process up to the time preceding t. If the process $X(t)$ is assumed to be *time homogeneous* and *Markov*, then $q_{ij}(t, F_t)$ is independent of t and F_t. For simplification, $q_{ij}(t, F_t)$ will then be written as $q_{ij}(t)$. Transition intensity for all possible transitions between states can be formed into a matrix $\mathbf{Q}(t)$, whose rows sum to 0, so that the diagonal entries are $q_{ii}(t) = -\sum_{j \neq i} q_{ij}(t)$. For multistate model in Figure 1, the transition intensity matrix is as follows.

$$\mathbf{Q}(t) = \begin{bmatrix} q_{11}(t) & q_{12}(t) & q_{13}(t) \\ q_{21}(t) & q_{22}(t) & q_{23}(t) \\ 0 & 0 & 0 \end{bmatrix}$$

Transition probability $P_{ij}(t)$ is the probability of being in state j at a time $t + s$ in the future, given the state at time s is i. If the process is assumed to be *Markov* then the conditional distribution of the future $X(t + s)$ given the present $X(s)$ and the past $X(u)$, $0 \leq u < s$, depends only on the present and is independent of the past.

$$P\left(X(t+s) = j \middle| X(s) = i, X(u) = x(u), 0 \leq u < s\right)$$
$$= P\left(X(t+s) = j \middle| X(s) = i\right)$$

Transition probability for all possible transitions between states can be formed into a matrix $\mathbf{P}(t)$ whose rows sum to 1. In general, the matrix $\mathbf{P}(t)$ is as follows.

$$\mathbf{P}(t) = \begin{bmatrix} P_{11}(t) & P_{12}(t) & \dots & P_{1n}(t) \\ P_{21}(t) & P_{22}(t) & \dots & P_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1}(t) & P_{n2}(t) & \dots & P_{nn}(t) \end{bmatrix}$$

The transition probability matrix can be obtained from the Kolmogorov equation as

$$\mathbf{P}(t) = e^{\mathbf{Q}t}.$$

In Jackson [7] the one-step transition probabilities in a discrete-time Markov chain can therefore be parametrized via

$$\mathbf{P} = e^{\mathbf{Q}}.$$

3. Hidden Markov Model

In a hidden Markov model (HMM) the states of the Markov chain are not observed. The observed data are governed by some probability distribution conditionally on the unobserved state. The evolution of the underlying Markov chain is governed by a transition intensity matrix **Q**. Hidden Markov models are mixture models, where observations are generated from a certain number of unknown distributions. However the distribution changes through time according to states of a hidden Markov chain, Jackson [6]. This figure gives an illustration for hidden Markov model.

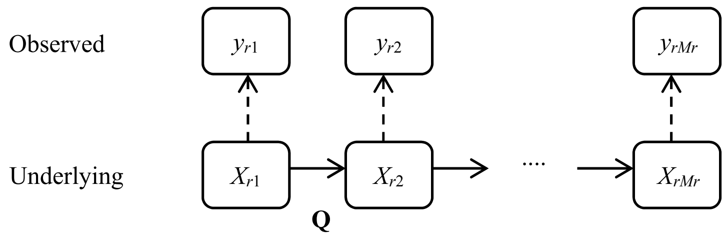

Figure 2 A hidden Markov model

Observation y_{rs} is the observed data for the r^{th} individual at the s^{th} observation, where $r = 1, 2, ..., N$, and $s = 1, 2, ..., M_r$. It can be either continuous or categorical data. The underlying state is denoted by $X_{rs} = i, \quad i = 1, 2, \ldots, n$. Probability distributions of y_{rs} conditional on the underlying state $X_{rs} = i$ are, $f_1(y|\theta_1), f_2(y|\theta_2), \ldots, f_n(y|\theta_n)$ where θ_k is a vector of parameters for the state i distribution. For example, if y_{rs} is Normally distributed for each state in the model, then

$$y_{rs} \left| \{X_{rs} = i\} \sim N(\mu_i, \sigma_i^2) \right.$$

$$f_i(y_{rs}|\mu_i, \sigma_i^2) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left[-\frac{(y_{rs} - \mu_i)^2}{2\sigma_i^2}\right].$$

The most efficient, and by far the most popular, approach for estimating the parameters of HMM is based on the Expectation-Maximization (EM) principle, Cappé *et al*. [2]. The EM algorithm is an iterative optimization method for maximum likelihood estimation of statistical models that involve unobserved (or latent) data. The application of EM to hidden Markov model is not straightforward and requires an additional inductive algorithm known as the *forward-backward*, introduced by Baum *et al.* [1]. In particular, EM is known to converge very slowly in some models. As an alternative, we used quasi-Newton optimization methods as well as on an efficient purely recursive algorithm for computing the log-likelihood.

Jackson *et al* [5] describe, individual r's contribution to the likelihood is

$$\begin{aligned} L_r &= P(y_{r1}, y_{r2}, \ldots, y_{rM_r}) \\ &= \sum P(y_{r1}, y_{r2}, \ldots, y_{rM_r} | X_{r1}, X_{r2}, \ldots, X_{rM_r}) P(X_{r1}, X_{r2}, \ldots, X_{rM_r}) \end{aligned} \tag{1}$$

where the sum is taken over all possible paths of underlying states $X_{r1}, X_{r2}, \ldots, X_{rM_r}$. Assume that the observed data are conditionally independent given the values of the underlying states.

$$P\left(y_{r1}, y_{r2}, \ldots, y_{rM_r} \middle| X_{r1}, X_{r2}, \ldots, X_{rM_r}\right) = P\left(y_{r1} \middle| X_{r1}\right) P\left(y_{r2} \middle| X_{r2}\right) \ldots P\left(y_{rM_r} \middle| X_{rM_r}\right).$$

Also assume the Markov property,

$$P\left(X_{rs} \middle| X_{r,s-1}, \ldots, X_{r1}\right) = P\left(X_{rs} \middle| X_{r,s-1}\right)$$

then

$$P\left(X_{r1}, X_{r2}, \ldots, X_{rM_r}\right) = P\left(X_{r1}\right) P\left(X_{r2} \middle| X_{r1}\right) \ldots P\left(X_{rM_r} \middle| X_{r,M_r-1}\right).$$

Decompose the overall sum in equation (1) into sums over each underlying state. The sum is accumulated over the unknown first state, the unknown second state and so on until the unknown final state.

$$\begin{aligned} L_r = &\sum_{X_{r1}} P\left(y_{r1} \middle| X_{r1}\right) P\left(X_{r1}\right) \sum_{X_{r2}} P\left(y_{r2} \middle| X_{r2}\right) P\left(X_{r2} \middle| X_{r1}\right) \ldots \\ &\sum_{X_{rM_r}} P\left(y_{rM_r} \middle| X_{rM_r}\right) P\left(X_{rM_r} \middle| X_{r,M_r-1}\right) \end{aligned} \tag{2}$$

From the above likelihood, there are several parameters that must be estimated, vector of parameters θ_i in the observation distribution, $f_i\left(y \middle| \theta_i\right) = P\left(y_{rs} \middle| X_{rs} = i\right)$, transition probability, $P_{ij}\left(t\right) = P\left(X_{rs} = j \middle| X_{r,s-1} = i\right)$ for $t = t_{rs} - t_{r,s-1}$, and initial state distribution $\pi_i = P\left(X_{r1} = i\right)$. Furthermore, the set of the parameters that must be estimated denoted by $\lambda = \left(\mathbf{P}, \boldsymbol{\theta}, \boldsymbol{\pi}\right)$ where **P** is transition probability matrix, $\boldsymbol{\theta} = \left(\theta_1, \theta_2, \ldots, \theta_n\right)^T$, and $\boldsymbol{\pi} = \left(\pi_1, \pi_2, \ldots, \pi_n\right)^T$.

The full likelihood function is,

$$L = \prod_{r=1}^{N} L_r$$

then log-likelihood function is

$$\log L = \sum_{r=1}^{N} \log L_r. \tag{3}$$

The first derivative of the log L, which in turn is written $\frac{\partial}{\partial \lambda} \log L$, will be searched by using the forward and backward variables. Instead of using the continuous transition probability, the transitions of a Markov chain are often characterized by a discrete transition probability matrix. Here's the definition of a forward and backward variables, Qin *et al*. [10]. For each r = 1, 2, ..., N, s = 1, 2,

..., M_r, and i, j = 1, 2, ..., n, forward variable is defined as follows.

$$\alpha_{rs}(i) = P(y_{r1}, y_{r2}, \ldots, y_{rs}, X_{rs} = i)$$
$$\alpha_{r,s+1}(j) = \sum_{i=1}^{n} \alpha_{rs}(i) P_{ij} f_j(y|\theta_j)$$

In matrix form

$$\boldsymbol{\alpha}_{rs} = \begin{bmatrix} \alpha_{rs}(1) \\ \alpha_{rs}(2) \\ \vdots \\ \alpha_{rs}(n) \end{bmatrix}.$$

For each r = 1, 2, ..., N, s = 1, 2, ..., M_r, and i, j = 1, 2, ..., n, backward variable is defined as follows.

$$\beta_{rs}(j) = P\left(y_{r,s+1}, y_{r,s+2}, \ldots, y_{rM_r} \middle| X_{rs} = j\right)$$
$$\beta_{rs}(i) = \sum_{j=1}^{n} P_{ij} f_j(y|\theta_j) \beta_{rs}(j)$$

In matrix form

$$\boldsymbol{\beta}_{rs} = \begin{bmatrix} \beta_{rs}(1) \\ \beta_{rs}(2) \\ \vdots \\ \beta_{rs}(n) \end{bmatrix}.$$

Matrix $\mathbf{B}_{rs}$ is defined as follows.

$$\mathbf{B}_{rs} = \begin{bmatrix} P(y_{rs}|X_{rs}=1) & 0 & \ldots & 0 \\ 0 & P(y_{rs}|X_{rs}=2) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & P(y_{rs}|X_{rs}=n) \end{bmatrix}$$
$$= \begin{bmatrix} f_1(y|\theta_1) & 0 & \ldots & 0 \\ 0 & f_2(y|\theta_2) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & f_n(y|\theta_n) \end{bmatrix}$$

so that the forward and backward variables can be expressed as a matrix multiplication.

$$\boldsymbol{\alpha}_{r,s+1}^{T} = \boldsymbol{\alpha}_{rs}^{T}\mathbf{PB}_{r,s+1}$$
$$\boldsymbol{\beta}_{rs} = \mathbf{PB}_{r,s+1}\boldsymbol{\beta}_{r,s+1}$$

Note that the likelihood in equation (2) can be written as the following matrix multiplication,

$$L_r = \boldsymbol{\pi}^T\mathbf{B}_{r1}\cdot\mathbf{PB}_{r2}\cdot\mathbf{PB}_{r3}\ldots\mathbf{PB}_{rM_r}\cdot\mathbf{1}$$

where **1** be a column vector consisting of 1s. The derivatives of the log likelihood can be formulated as

$$\frac{\partial\log L_r}{\partial\pi_i} = f_i\left(y|\theta_i\right)\hat{\beta}_{r1}\left(i\right)$$

$$\frac{\partial\log L_r}{\partial P_{ij}} = \sum_{s=1}^{M_r-1}\hat{\alpha}_{rs}\left(i\right)\hat{\beta}_{r,s+1}\left(j\right)f_j\left(y|\theta_j\right)$$

$$\frac{\partial\log L_r}{\partial\theta_j} = \pi_j\hat{\beta}_{r1}\left(j\right) + \sum_{s=1}^{M_r-1}\hat{\beta}_{r,s+1}\left(j\right)\sum_{i=1}^{n}\hat{\alpha}_{rs}\left(i\right)P_{ij}$$

Where $\hat{\alpha}_{rs}\left(i\right)$ and $\hat{\beta}_{r,s}\left(j\right)$ are the scaled forward and backward variables.

Derivatives of the full likelihood is,

$$\frac{\partial}{\partial\lambda}\log L = \sum_{r=1}^{N}\frac{\partial}{\partial\lambda}\log L_r.$$

To find estimators $\hat{\lambda}$ that maximize log L, we used quasi-Newton method.

4. Health Insurance

Health insurance is an insurance product that provides certain benefits if the insured suffered illness, accident, or receive medical care. Health insurance product consist of social and commercial health insurance, Thabrany *et al*. [13].

Individual health insurance premium is the amount of money that must be paid by the person as an insured or policyholder in return for a guarantee of cost as a result of the onset of a disease risk as stated in the policy. Meanwhile, health insurance premium are generally set at the sum of the individual premium in one agency.

This paper discuss about an insurance product that can be organized by small-scale medical institutions such as outpatient clinics. This insurance covers the members when he fell ill and went to the medical institutions concerned. Members only need to pay insurance premium once at the beginning of the policy. Subsequently the patient does not have to pay again if he gets a medical examination within the policy period. If the policy period had expired, members can extend their health insurance by paying premium again, and so on.

Suppose the stochastic process $\{X_s, s = 0,1,\ldots,M\}$ is a discrete time Markov chain. At time *s*, the risk is in state *h*, $X_s = h$. Health insurance premium for the occurrence of transition from state *i* to state *j* with the benefit of 1 is as follows.

$$\overline{A}_{hij}(t) = \sum_{s=0}^{t} P_{hi}^{t} q_{ij} v^{t} \tag{4}$$

Where,

P_{hi}^{t} is *t*-step transition probability from state *h* to state *i*,

q_{ij} is transition intensity from state *i* to state *j*,

v^{t} is present value of of a payment of 1 that will be expired in the coming period *t*, with $v = (1+i)^{-1}$, and *i* is the interest rate.

5. Application

The data used is the patient visit data in Polyclinics Cihideung Garut West Java between January 2012 to April 2012. The data is taken from a sample of 200 patients. The information used is the patient ID number, time of visit, and cost.

The model that will be used is the multistate model with 4 states. The states show the patient's health condition,

1: healthy
2: light disease
3: moderate disease
4: severe disease.

Time unit used is the week. Informations regarding the time of recovery are not available in the data because patients only visit when sick so assumed if the patients do not visit again within one week then the patient is considered cured (healthy). Status of the patient journey from a state to another state is assumed to follow a Markov assumption. Direct transitions are possible from healthy state to diseased state and vice versa as illustrated in the following figure.

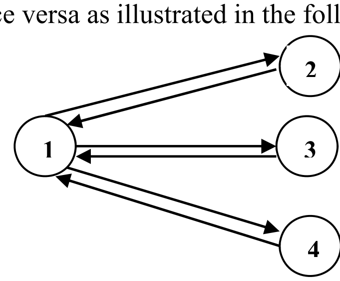

Figure 3 Multistate model for patient's visit data

Based on the picture above the transition intensity matrix can be arranged as follows,

$$\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{21} & q_{22} & 0 & 0 \\ q_{31} & 0 & q_{33} & 0 \\ q_{41} & 0 & 0 & q_{44} \end{bmatrix}$$

where $q_{ii} = -\sum_{j \neq i} q_{ij}$, $i, j = 1,2,3,4$.

On patient visits data, there is no information about the type of the diseased, but it is assumed there is a relationship between the type of diseased with cost. Thus, in this model it is assumed the state of patients was not observed the cost can be observed.

Patient's cost is assumed Normally distributed. If y_{rs} shows the cost to the r^{th} patient on the s^{th} visit, then the observation probability distribution can be written as follows,

a. $y_{rs} | \{X_{rs} = 1\} \sim N(\mu_1, \sigma_1^2)$,
b. $y_{rs} | \{X_{rs} = 2\} \sim N(\mu_2, \sigma_2^2)$,
c. $y_{rs} | \{X_{rs} = 3\} \sim N(\mu_3, \sigma_3^2)$,
d. $y_{rs} | \{X_{rs} = 4\} \sim N(\mu_4, \sigma_4^2)$.

When the patient is health then the costs equal to zero, so that $\mu_1 = 0$ and $\sigma_1 = 0$.

Parameter estimation is done with the help of library msm version 0.7 on R software. In the model described earlier, there are 12 parameters to be estimated, q_{12}, q_{13}, q_{14}, q_{21}, q_{31}, q_{41}, μ_2, μ_3, μ_4, σ_2, σ_3, and σ_4. Estimation process is an iterative process requiring initialization values for each parameter. The initialization value can be determined based on the literature, previous research, or determined by the researcher. In here the initialization of transition intensity is as follows.

$$\mathbf{Q}_0 = \begin{bmatrix} -0,4 & 0,1 & 0,1 & 0,2 \\ 0,5 & -0,5 & 0 & 0 \\ 0,6 & 0 & -0,6 & 0 \\ 0,8 & 0 & 0 & -0,8 \end{bmatrix}$$

Initialization values for the parameters in the probability distribution of the observations is μ_2 = Rp. 33.000,00, μ_3 = Rp. 48.000,00, μ_4 = Rp.87.000,00, σ_2 = Rp. 8.000,00, σ_3 = Rp. 5.000,00, dan σ_4 = Rp. 17.000,00.

By using initial values above, obtained estimates for the transition intensities as follows.

$$\hat{\mathbf{Q}} = \begin{bmatrix} -0,4453 & 0,03283 & 0,1444 & 0,2681 \\ 0,5742 & -0,5742 & 0 & 0 \\ 0,6135 & 0 & -0,6135 & 0 \\ 0,8222 & 0 & 0 & -0,8222 \end{bmatrix}$$

Estimated values for the parameters in the probability distribution of the observations are,

$$\begin{aligned} \hat{\mu}_2 &= \text{Rp.}33.000,01, \quad \hat{\sigma}_2 = \text{Rp.}17.010,76, \\ \hat{\mu}_3 &= \text{Rp.}48.000,00, \quad \hat{\sigma}_3 = \text{Rp.}11.030,10, \\ \hat{\mu}_4 &= \text{Rp.}86.999,94, \quad \hat{\sigma}_4 = \text{Rp.}28.719,32. \end{aligned}$$

Based on estimates of the transition intensity, in the first row can be seen that the risk of a healthy patient will suffer severe disease is greater than light and moderate diseases. The second, third, and fourth row show that the cure rate of patients suffering from severe disease is greater than light and moderate.

Parameter estimation results in the probability distribution of the observations show that the average cost of patients suffering from light disease is Rp. 33000.01 with a standard deviation of Rp. 17010.76. The average cost of patients who suffer moderate disease is Rp. 48000.00 with a standard deviation of Rp. 11030.10. The average cost of patients suffering from severe disease is Rp. 86999.94 with a standard deviation of Rp. 28719.32.

Estimation of transition probabilities in a one week period are as follows.

$$\hat{\mathbf{P}} = \begin{bmatrix} 0,7383 & 0,0208 & 0,0897 & 0,1512 \\ 0,3635 & 0,5688 & 0,0247 & 0,0430 \\ 0,3812 & 0,0060 & 0,5675 & 0,0453 \\ 0,4636 & 0,0075 & 0,0327 & 0,4962 \end{bmatrix}$$

In the above transition probability matrix, the first row shows that the probability of healthy patients staying healthy in one week is greater than falling ill. Row 2, 3, and 4 show the probability of a patient will recover within one week is less than the probability of stay in the same diseased.

Premium value is determined by equation (4). The required cost when patients suffer light, moderate, and severe diseases are assumed constant and determined as $\hat{\mu}_2 \approx \text{Rp.}33.000, \hat{\mu}_3 = \text{Rp.}48.000, \text{ dan } \hat{\mu}_4 \approx \text{Rp.}87.000$. Multistate model used assumes that the direct transitions can occur between states 1↔2, 1↔3, dan 1↔4, so that the premium calculation is done based on the values of $\hat{P}_{11}^t, \hat{q}_{12}, \hat{q}_{13}$, and $\hat{q}_{14}$ with discrete time approach,

$$33.000\sum_{s=0}^{t} v^t \hat{P}_{11}^t \hat{q}_{12} + 48.000\sum_{s=0}^{t} v^t \hat{P}_{11}^t \hat{q}_{13} + 87.000\sum_{s=0}^{t} v^t \hat{P}_{11}^t \hat{q}_{14}$$

where $\hat{P}_{11}^{t}$ is the elemet of $\hat{\mathbf{P}}^{t}$ in first row and first column, and $\hat{q}_{12}, \hat{q}_{13}, \hat{q}_{14}$ respectively are the element in first row column 2, 3, and 4 of the matrix $\hat{\mathbf{Q}}$, also t in week.

If it is assumed that the interest rate is 0%, the health insurance premiums is as follows.

a. Premium value for half-year period is Rp.480.897,00.
b. Premium value for one year period is Rp.895.467,40.
c. Premium value for one half-years period is Rp.1.256.972,00.
d. Premium value for two-years period is 1.572.204,00.

References

[1] Baum, L. E. dan Petrie, T., 1966, Statistical Inference for Probabilistic Functions of Finite State Markov Chains, *Ann. Math. Statist*., 37, 1554–1563.

[2] Cappé, O., Buchoux, V., and Moulines, E., 1998, Quasi-Newton Method for Maximum Likelihood Estimation of Hidden Markov Models, *IEEE on Acoustics, Speech, and Signal Processing, ICASSP98*, 4, 2265–2268.

[3] Chen, H. H., Duffy, S. W., and Tabar, L., 1996, A Markov Chain Method to Estimate the Tumour Progression Rate from Preclinical to Clinical Phase, Sensitivity and Positive Predictive Value for Mammography in Breast Cancer Screening, *The Statiscian*, 45, 3, 307-317.

[4] Durbin, R., Eddy, S., Krogh, A. dan Mitchison, G., 1998, *Biological Sequence Analysis: Probabilistic Models of Proteins and Nucleic Acids*, Cambridge University Press, Cambridge.

[5] Jackson, C. H., Sharples, L. D., Thompson, S. G., Duffy, S. W., and Couto, E., 2003, Multistate Markov Models for Disease Progression with Classification Error, *The Statistcian*, 52, 193–209.

[6] Jackson, C. H., 2006, Multistate Modelling with R: the msm package, http://rss.acs.unt.edu/Rdoc/library/msm/doc/msm-manual.pdf, accessed 6th March 2013.

[7] Jackson, C. H., 2011, Multi-State Models for Panel Data: The msm Package for R, *Jornal of Statistical Software*, 38.

[8] Juang, B. H. dan Rabiner, L. R., 1991, Hidden Markov Models for Speech Recognition, *Technometrics*, 33, 251–272.

[9] Meira-Machado, L., 2011, Inference for Non-Markov Multi-state Models: An Overview, *REVSTAT*, 9, 1, 83-98.

[10] Qin, F., Auerbach, A., and Sachs, F., 2000, A Direct Optimization Approach to Hidden Markov Modeling for Single Channel Kinetics, *Biophysical Jornal*, 79, 1915–1927.

[11] Ross, S. M., 2010, *Introduction to Probability Models*, 10th edition, Elsevier, Oxford.

[12] Satten, G. A. dan Longini Jr, I. M., 1996, Markov Chains with Measurement Error: Estimating the 'True' Course of a Marker of the Progression of Human Immunodeficiency Virus Disease, *Applied Statistics*, 45, 3, 275-309.

[13] Thabrany, H., Surachmad, S., Iskandar, K., Handayani, Nurhayati, Hidayat, B., 2005, *Dasar-dasar Asuransi Kesehatan Bagian A*, PAMJAKI, Jakarta.

THEORETICAL METODOLOGY STUDY BETWEEN MSPC VARIABLE REDUCTION AND AXIOMATIC DESIGN

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Abstract. This paper proposes a variable ratio reduction methodology Multivariate Statistical Process Control (MSPC) and Axiomatic Design (AD). The benefit of the proposed method is to reduce the number of variables that must be measured, thus reducing the time and costs associated with the inspection. MSPC method of dimension reduction approach is to choose a variable process by maintaining as much information about the full set of variables wherever possible, while the dimension reduction AD approach is based on a series of work processes and customer needs. This paper examines the concept of MSPC by Gonzales and Sanchez (2010) and AD theory of Suh (2001) developed by Brown (2005). The findings yield advantages and disadvantages from the point of view of Statistical Quality Control, the two methods are complementary approaches can be used in solving the case of a business process. Preliminary selection of variables is performed using MSPC due to large data to be screened without sacrificing the quality of the information. AD is then used in the next screening process to come up with most significant variables for service quality business process analysis in air transportation.

Key words: variables reduction, variables weight, MSPC, Axiomatic Design.

1. Introduction

The main objective of statistical process control (SPC) is to maintain a stable process that will produces minimum variability. Monitoring is not difficult in case only single variable is observed. On the contrary, problem may arise if the observations were made in the multi-variable process. To further facilitate the process monitoring, it is necessary to conduct selection of significant variables (as representative of whole variables) without impair the existing/quality of information. Disposing or reducing number of variables could imply risk of reducing the effectiveness of process monitoring in detecting out-of-control (OOC) situations. In cases where there are doubts about eliminating some variables, the decision could be delayed until a set of OOC situations is available, the analysis of which can help to reveal whether variables can be disposed off without losing critical information. Variable selection has been an important issue in many areas of both theoretical, and applied statistics. The selection of variables for SPC should always follow some engineering criteria according to the functionality of the part.

Based on some previous research, MSPC approaches consists of two groups. The first grup which is procedures that rely on the multiple correlation coefficients between the selected variables and the discarded ones. For instance, Beale,et al.(1967) select the *p* variables using the interdependence analysis. This method, selects the *p* variables that maximize the minimum multiple correlation coefficient between the *p* variables and each of the *K-p* discarded ones (minimax),

analyzed were B = $\frac{K!}{p!(K-p)!}$ where K is initial variables and p is variables selected. Beale's opinion has the disadvantage that when K is large the cost of observation will be high.

Jolliffe (1972) introduced the stepwise method to overcome these weaknesses. In each step, the variable with the largest multiple correlation coefficients (compare with the remaining ones) is discarded because it is the variable with the more redundant information. Because these selection methods are based on correlation, the solution does not depend on the scalling of the variables. A more efficient approach is proposed in Mc.Cabe (1984). He introduced a concept of "principal variables" as the *p* variables that contain as much sample infomation as possible, measured by the conditional covariance matrix of the *K-p* variables given the selected *p*, denoted as $\Sigma_{K-p.p}$. This method could also be applied using the correlation matrix of data instead of the covariance matrix. By doing so, the selection does not depend on the scaling of the variables. One common difficulty of all these methods is that the number *p* of selected variables must be chosen. Beale, et al. (1967) proposes obtaining the value *p* as a function of the multiple correlation coefficients. For Jollife's methods (1972, 1973) proposes setting *p* equal to the number of principle component (PCs) needed to account for some proportion of the total variability. McCabe (1984) proposes choosing the value *p* according the percentage of variation explained for the selected subset of *p* variables.

The latest findings on variable selection proposed by Gonzalez and Sanchez (2010) produced some pluses and minuses. The plus is statistical quality control procedures that rely on the parameter estimates and the level of importance of each variable. Whereas the only drawback is that the variables selection is only made from the viewpoint of statistical techniques.

On the other hand it is necessary that the selection of these variables be done by taking view from both standpoint of statistical and engineering process respectively. This paper proposes an approaches, of which the weakness of their findings can be reduced with the Axiomatic Design (AD) method which is based upon the approach of process engineering. By using experienced technicians who can provide information about potential error of the available data, we will avoid the risk of reducing the effectiveness of the monitoring process by removing the useful variables. In this paper, the criteria of the variable selection in SPC is following variables selection in process engineering functions, that is the observed quality characteristic is the most influential variable in the process. The proposed methodology is the selection of variables for measuring customer satisfaction on service quality of airlines industry, refering to the papers from Gonzales and Sanchez (2010) as well as the processed data from Triwidiastuti (2006).

2. Main Results

This article proposes a methodology for the selection of variables that have several advantages compared with existing theories, as presented in Figure 1.

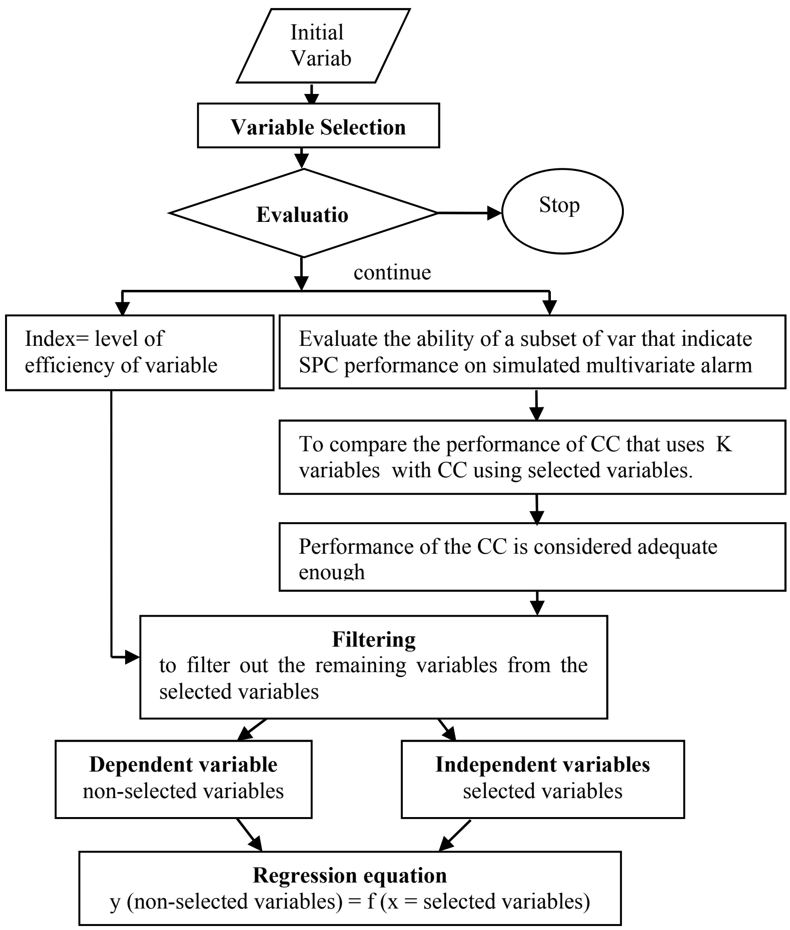


Figure 1. The proposed methodology for the selection of variables by Gonzales and Sanchez (2010)

This proposed methodology Gonzales and Sanchez (2010) in Figure 1 consists of three steps: (a) variable selection, (b) evaluation, and (c) filtering. These steps are run in iterations until it is completed at a predetermined stage. The first round, step (a) selecting a single variable that contains the greatest amount of information from the initial variable K. They proposed procedure, based on an oblique rotation of the independent factor obtained by PCA (Principal Component Analysis). The next step, (b), is to evaluate the existing information in the variable that has been selected in the previous stage, with aims to determine whether the selection could stops or be continued. Two proposed criteria for step (b) is:

- The first criteria, a subset of selected variables were evaluated based on the total amount of information contained. They specify the index that measures the information, which can indicate the level of efficiency of these variables. As a complementary tool, they propose the index of benefits along with a general index that measures the information in the selected variables.

- The second evaluation criteria, based on Principal Alarm Analysis (Sanches, 2006), used to evaluate the ability of a subset of variables that indicate SPC performance on simulated multivariate alarm. The simulated alarm is based on a shift in direction towards the principal component of a controlled data set. The second criterion is to compare the performance of control chart (CC) that uses all the variables K with control chart (CC) using selected variables. If the performance of the control chart is considered adequate enough in the evaluation stage, the selection is stopped.

The next step (c) is to filter out the remaining variables from the selected variables. At this stage, they form a regression equation with the non-selected variables as the dependent variable and the selected variables as independent variables so y (non-selected variables) = f (x = selected variables). In the first iteration, the results of step (c) is a *K-1* data set that has been filtered. When they repeat the iterative procedure will yield in a new variable selected, so that eventually create more and more number of selected variables. At the end of the procedure, they will have a subset of the variables $p \leq K$.

Variable-Selection Methodology

The goal of this selection step is to determine, at each iteration *t,* the variable that contains the largest amount of information. In order to decide how to measure the information in a set of variables, we will constraint to the linear relationships among them. Then, the in-control covariance Σ_K (or correlation) matrix will be used as a measure of their joint information. In the first iteration of the method, the selection step is performed with all the *K* variables. Once a variable is selected using the statistical technique explained below, the next selection is performed with the remaining *K* -1 nonselected variables, but filtered out from the information of the selected one. This filtered dataset of *K*-1 variables obtained at the end of the first iteration will be denoted as $X^{(1)}$. At the beginning of the t iteration *t*-1 variables have already ben selected, so that $S_{(t-1)} = (X_1, X_2, X_3 ... X_{t-1})$ where X_i is the variable seleced in *i* iteration; X^{t-1} is the selection step of this iteration; with the $k = K — t + 1$ remaining variables, but filtered out from the information of the variable subset $S_{(t-1)}$. The covariance matrix of X^{t-1} is denoted as $\Sigma_k^{(t-1)}$ which is the covariance of the *k* variables conditioned on the selected ones.

Gonzales dan Sanches (2010) proposed to select the variable at each iteration is based on PCA and be labelled as the OR method because it be based on oblique rotations. The main goal of PCA is to project a set of dependent variables X_1, X_2, X_3,.... X_k into a set of independent PCs Y_1, Y_2, Y_3,... Y_K. PCA is based on the singular value decomposition (SVD) of Σ_k and holds

$$\Sigma_K = C_K L_K C_K'$$

with the corresponding eigenvalues λ_j, j=1,2,...K and SVD still exists in the

related with only one variable, but all the elements in c_j might have similar values without a dominant one. Also, a variable can have high loads in several eigenvectors, which may lead to a poor selection capacity. To avoid this drawback, the proposed OR method uses some rotation of these eigenvectors, by using promax oblique rotation. The rotation is performed so that each new vector is more clearly associated with a single variable or a subset of highly related variables. The promax solution is achieved by first rotating to an orthogonal varimax solution, and

matrix $V_K^{\mathrm{var}i}$ with vectors $v_j^{\mathrm{var}i}$, which have a large number of loadings $v_{ij}^{\mathrm{var}i}$ with

After varimax rotation, each new rotated component $Y_j^{\mathrm{var}i}$, j = 1, 2, ...K, tends to be

clusters of variables). The matrix V_K^{pro} is usually denoted in the factor-analysis literature as a pattern-loadings matrix. The pattern loadings can be interpreted as measures of the unique contribution that each rotated component makes to the variance of the original variables. Consequently, the promax solution is a better tool for variable selection than using simple PCA.

At each iteration, the variable that contains the largest amount of information of the

selection should not only take into account the individual loadings v_{ij}^{pro}, but should

large absolute values of v_{ij}^{pro}, that component will contain a large amount of

component with maximum q_j is determined, the selected variable will be the one with the largest absolute loading in that component. It is interesting to note that the

...e too that, at each iteration $t > 1$, this OR selection criteria is performed using Σ_K^{t-1}.

Definition 2.1.

Evaluation Step

Two general approaches proposed by Gonzalez and Sanchez (2010) is to evaluate the subset of selected variables. In the first approach, the selected variables are evaluated as a function of the total amount of information contained in them. This evaluation approach is not target oriented in the sense that it only evaluates the ability of the selected subset to reproduce the general features of the complete data, irrespective of the specific use of the variables by the analyst. In this evaluation approach, we define two R-like indices that measure the amount of information contained in the selected subset of variables. Conversely, the second evaluation approach principal alarms analysis (PAA) is target oriented. The subset of selected variables is evaluated according to their ability to perform an efficient statistical monitoring of a multivariate process. This second approach, which is more time consuming, is based on the simulation of some out-of-control situations using the method of PAA. Evaluation step in this paper use first approach, two R-like indices because the selected variables are evaluated as a function of the total amount of information contained in them.

This paper uses the Axiomatic Design (AD) approach to evaluate the subset of selected variables, this design method was originally introduced by Suh (2001). AD is a structured approach to performance improvement in complex systems at the stages of concept, at preliminary or basic design and at detailed design phase of systems engineering models. This analytical and structured method links functional requirements for the product system with process design parameters. If there is a change in the design parameters, AD quantitative methods can be used to assess the sensitivity of the functional requirements in a particular process. AD techniques reduce the risk incidences, reduce costs and speed up the process for the products to each market / customer, with multiple stages (Suh, 2001). Variable reduction process using AD has been done by Triwidiastuti (2010) to determine the factors critical to quality which are used then to measure the service quality of Higher Education.

Original AD concept developed by Suh (2001) is aim to reduce the incidence of risk, reduce costs and speed up processes of product to market / customers by:

- Develop and incorporate concepts of process design into a continuous and measurable activities tailored to specific needs
- Communicate the plan to all stakeholders before technical drawings (Computer Aided Design) are prepared and documented.
- Improve the design quality by analyzing and optimizing the design architecture.
- Track and define the customer wants accurately and incorporate such requirement into detailed design specifications
- Document and communicate the design logics(how and why) clearly
- Identify design problems at early stage and complete the cycle: design-build-test-redesign with minimal costs.
- Allow management to observe the interrelationship/dependencies among design structures, to optimize the scheduling, identification and reduction of risks

Axiomatic process can result in:

- detailed description of the functions of an object (usually derived from customer needs desires).
- description of the object that will fulfill that functions.
- description of how the function will be fulfilled.

Axiomatic approach

This approach has been widely used for the design (both at initial design and at redesign) of products, its processes and organizations starting from describing or defining the problem or customer needs. Problem or customer needs definitions are often subjective because of differences in interpretation by different groups or individuals. Difficulties will arise towards the end of the design because of differences in interpretation, therefore the problem or customer requirements should be described as detailed and as accurate as possible at the earliest stage, the details of which should have a physical operational features. AD concept helps the designer to specify and incorporate such design into each phases of activity to suit the customer's needs. Designers can use the design tool to produce a more efficient and better design than before (Triwidiastuti, 2010).

There are four main concepts in AD, that are: Domain, Hierarchies, Zigzagging and Design axioms (Brown, 2005).

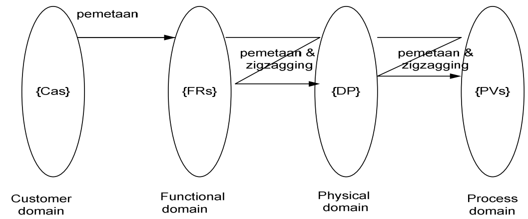
a. Domain

Contains specific requirements that are mapped at design phase in the form of characteristic parameters.

Tabel 3. Axiomatic Design Domain

Design Stage	Design Domain	Design Element
Design Concept	customer	- Customer needs (CNs), benefit/values customer seeks - Customer needs , identified and formulated into functional forms
Product design	functions	- Functional description (FRs) of the solution - Constraints (Cs)
Process design	physical	- Able to cater functional requirement - Design parameter (DPs) solution alternatives - Plans were formulated into a draft
	process	Variables/process atributes

The basic concept of this domain consists of 4 stages:



For each pair of domains, left domain is "what we want to achieve" while right domain is a "how to achieve it". Each domain means:

Customer	: Advantages / benefits desired by customers
Functional	: Functional requirements of the design
Features/Physical	: Design parameter
Process	: Variables / process attributes

Tabel 4. Axiomatic Design Notation

Functional Requirements (FRs)	is an independent set of minimum requirements that can complement the characteristics of the functional requirements of the design at a functional domain
Constraint (Cs)	constraint is limit of resolution / acceptable design
Design Parameters (DPs)	is an element of the design of the physical domain chosen for a particular FRS
Process Variables (PVs)	is an element in the process domain that shows characteristics of design parameters

Tabel 5. Examples of Axiomatic Design Application

Type of activities	Domain			
	customer	functions	features	Process
Food preservation	durable food	refrigerate food	refrigerator	how to make a refrigerator (in detail)
	what →	how what →	how what →	how

Relationship mapping each domain can be in a form of a matrix (between FRS and DPs, between DPs and PVs).

	DP1	DP2	DP3	DP4
FR1	X	O	X	O
FR2	X	X	X	O
FR3	X	O	X	O
FRn	X	X	O	X

Where the notation X indicates a correlation and notation O as uncorrelation, and each cell shows the mathematical relationship between FR and DP.

b. Hierarchies (hirarki)

The 2nd axiomatic design concept is HIRARCHIES that are represented by a hierarchical design (design architecture). Starting from the highest level designer will choose the most suitable design and then decompose the FRs starting from the highest to the lowest FR. Highest FRs is paired with the highest DPs. This decomposition produces the appropriate level layer between FRS and DPs, performed at each level one by one. The same decomposition also conducted on a pair DPs and PVs, and for each level as well.

c. Zigzagging

The 3rd of axiomatic design concept is zigzagging that describes the design decomposition processes into hierarchies and pairing inter-domain levels. The designer follows the pattern domain "what = what" and "how = how" for each level of the hierarchy.

d. Design axioms

Axioma 1: *Independence Axioma*

Part of the design that can be separated (separable) so that changes in one of such separate part will not or minimize the effect to the other part as minimum as possible in other words maintain independence between FRS and DPs DPs received on the draft. Each DP is set by the FRS FRS without involving others.

Axioma 2 : *Information Axiom*

Minimize information: among design alternatives that fit with Axioma 1, the best is to keep information to minimum and which will give maximum probability of success. Cotoia and Johnson (2001) divides this type into three (3), ie uncoupled, coupled and decoupled. Design that is not comply with the independence Axioma is called coupled. This design occurs in two conditions, namely when there are fewer DPs than the FRS (DPs<FRs), so several FRS affects 1 DPs. Otherwise 1 DPs affect some FRS. Examples of this design is the water faucet, because each DP control/influence 2 FRs (regulate temperature and water flow). This can be explained as follows: $FR_1 = DP_1$ and DP_2; $FR_2 = DP_1$ and DP_2 The opposite is also $DP_1 = FR_1$ and FR_2; $DP_2 = FR_1$ and FR_2 so that

$$[A] = \begin{pmatrix} A_{11} & A_{12} & 0 \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix}$$

The design which comply with independence Axioma are called uncoupled or decoupled. The design is called uncoupled if each DP is independent, only affects the 1 (one) FRs, does not affect other FRs. For example FR1 affect DP1, DP2 affects FR2, FR3 affect DP3. Uncoupled characteristics are all non-diagonal elements equal to zero. $\{FR\} = A\{DP\}$ with m = n = 3 where $FR_1 = A_{11}DP_1$, $FR_2 = A_{22}DP_2$, and $FR_3 = A_{33}DP_3$

$$|A| = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}$$

As for the design of decoupled occurred on 2 conditions:

- first condition if 1 DPs affecting 2 or more FRS or FRs > DPs. This problem can be solved by increasing the qualified DPs.
- the second condition if DPs is enough but 1 or more FRS depends on one or more DPs. This is expressed in matrix in the form of m x n

$$[A] = \begin{pmatrix} A_{32} & 0 & 0 \\ A_{22} & A_{23} & 0 \\ A_{12} & 0 & A_{11} \end{pmatrix}$$

The data processing

The limitation of this model is data with finite positive matrices, minimum interval scale, not in mass service industry, data in discreet and base on small subgroup observation. Assumption in model development: variables/attributes as result of reduction are able to represent the whole quality characteristics being observed, as they represent most important variables/attributes from customer side of view (Triwidiastuti, 2006).

The data from research made by Gonzales and Sanchez (2010) is a tangible data, that is manufacturing data of production process of the window frame for the vehicle door. While the proposed research data is an intangible customer satisfaction data in form of ordinal scale. This data is the measurement of customer perception of service quality of airline company. because of the perception variables are difficult to measure quantitatively measured, then the solution is the perception variables measured by ordinal scale later in the transformation.

In Triwidiastuti (2006), the determination of the variable refers to some literature on airlines services (Banfe, 2001), quality of services (Ramaswamy, 1996; Holloway, 2002) and a standard operating procedure (SOP) PT. Garuda Indonesia. Questionaires to the respondent are based on the criteria: a frequent flyer passengers, traveling by plane at least 5 times a year, and short-haul flights (Banfe, 2001). Following the theory of expert system then the questionnaires were given to limited number of respondents that were carefully chosen and selected. Quantitative measurements were performed for customer satisfaction reveal requirement, since reveal requirements are more easily measured. Questions are divided into 3 parts, namely pre-flight, in-flight and post-flight, and generate 41 variables. These variables are grouped into 2 categories: product quality variables (11 variables) and quality of service (30 variables). Study variables are presented in Table 6.

Table 6. Research Variable

Product quality	safety	quality of service	Reservation officer serving with friendly and courteous
	security		Location reservations can be reached / contacted easily
	insurance		Speed the reservation process, to obtain confirmation
	frequency		Condition check-in counter (location, equipment, clarity signage)
	timing		Cleanliness waiting room
	flight connection		Submission of information waiting room with clear sound
	punctuallity		Explanation officer for cancellation, delay and withdrawal time flight delivered immediately
	airport location accesability		Skilled personnel to handle disruptive or abusive passengers
	seat accesability		The ability and professionalism of officers (handling an efficient check-in): service time, flexible on certain requests, special requests confirmation of passengers carried on reservationHow to accept the passenger cabin crew during boarding (smile and friendly)
			Knowing passenger profiles (type of diet, preferred seating, etc.)
			How officers handle the lost ticket (left behind ticket)
			Explanation for the special circumstances attendant baggage: excess baggage, baggage size is too big, too

			heavy baggage
			Availability of special counters for check-in (without luggage, faster time, re-check-in, check-in group, family check-in, check-in counters for long-distance, baggage counter to check-in remotely)
			Special handling check-in: VIP, CIP, F / C Class Executive card holders, transit without visa, passengers with less limitations / should be guided (incapacitated pax), passengers with extra body shape (tall, fat, large), with a passenger seat wheel, passengers who are pregnant, oxigen bottles, prisoners, passengers in conditions deported
			Explanation officer for passengers who would go on to change planes (procedures and advanced flight information)
			Handling passenger name that does not exist in the list of attendees
	benefit for loyal customer		Explanation officer for cancellation, flight delay and withdrawal time
	airport quality service		Cabin crew welcome greeted politely and ask the passenger seat
			greet and welcome politely cabin crew and ask the passenger seat
			Cabin crew always help passengers find their seats and gave instructions that the luggage must be stored so as not to disturb other passengers and the safety of flight
			How to dress neat, clean, well maintained and in harmony with body size
			Well-timed in handling; 5 mintes before the time departure, the plane's door is locked, so that the engine can be turned on and take off as scheduled.
			Announcement: delay of the flight has to be informed to the passanger as soon as possible.
			Announcer has a clear intonation and profesional.
			Safety demostrastion is done with actual clean kit; demonstrated in profesional manner.
			Safety and security kit check (seat and seat belts) before take off and during the flight.
			Polite and professional communication
			Professional, efficient, curteous and friendly manner.
			Helpful and aware of those who might need help.
			Seat and seat belt checking n professional manner

60 pieces of questionnaire were distributed, 56 pieces were returened and as many as 51 pieces can be processed. Ordinal scale is used (discrete) transformed into interval scale (continuous) with Multidimentional Scaling and using MATLAB software. Cronbach α is used for Data processing for reliability test of respondents, done for one measurement (one shoot) only. Quality of products variable produced α of 0,849, while for service quality α of 0.967. So it can be concluded that the measuring instrument is reliable. Validity test conducted by Principle Component Analysis. The data variables were selected according to the stages proposed by Gonzales and Sanchez (2010), yielding 30 variables. This variables then selected again using AD methods.

Field findings indicate that the proposed model is able to measure customer satisfaction and measuring quality caracteristic of services (gap 3). Problems arisen during field application is in getting interval data to gauge customer perception. This is solved by statistic manipulation, which is data transformation.

3. Concluding Remarks

Table 7. State of The Art SPC variable reduction metodology of Gonzales dan Sanches (2010) and Brown (2001).

Procedure / Step	**MSPC by Gonzales and Sanches**	**AD by Brown**	**The Data Processing**	**Advantages**	**Disadvantages**
Evaluate the subset of selected variables.	PAA	Qualified technician experience and brain storming	Measurement variables are characteristics of quality	MSPC based on OOC, while AD considering the characteristic process engineering	MSPC not consider characteristics of the process. AD does not consider the process variability
Significant level of variable	Eigen value and eigen vector	Brain storming, based on customer requirements and process quality standard	Matrix data	Eigen value β_i to describe the variability component i, which is the most important component is indicated by the maximum variability. AD based on customer requirement point of view	
Variable correlation	Correlation in matrix	zigzaging	Matrix data	-	-
Variabilitas inter dan antar variabel	Covariance matrix	-	Matrix data	MSPC, the covariance matrix focuses on precision and accuracy of data	MSPC, the covariance matrix focuses on precision and accuracy of data
Reducing variables	Performance SPC, based on process capability	Variables that can be accepted as critical to quality (CtQ)	CtQ matrix	AD: business processes can be mapped more optimal. MSPC: fewer observations MSPC and AD: a more efficient time and cost	

The conclusion of this paper is

1. Gonzales and Sanches (2010) method approach results in large variabiity of simulated CC conditions in the in controll data, this situation may happen in some actual processes but may not occur in other processes, which mean that the selected variables may not result in proper detection of OOC condition.

2. The approach of Gonzales ad Sanchez (2010) is that the proposed method is based on statistical procedures that rely on the parameter estimates. Therefore, it is advisable to take into account the uncertainty due to sampling estimates using re-sampling techniques. This paper however do not use resampling techniques

3. Another drawback of their approach is the possibility of a variable that contains useful information is lost in the process of selection variables. This drawback can be reduced with AD approaches, namely the selection of variables based on engineering processes. It is suggested that these results should be reviewed by a qualified techniciann who can provide more in-depth information as well as potential error in the available data. So that we maybe able to avoid the risk of reduced effectiveness of the monitoring process for wasting useful variables.

4. The strength of MSPC method is that the resulting matrix is a representation of a covariance matrix, where the covariance matrix is a matrix that describes the variability of each and between variables / attributes. This matrix can also be used to determine the weight of the components observed variables (characteristics of quality, QC) which are considered some of the most important QC aspect.

Acknowledgement.
My knowledge in the field of Process Control is enriched with knowledge gained from Gonzales and Sanchez (2010) and Brown (2005) that complement each other.

References

[1] Brown, C.A., 2005, Teaching Axiomatic Design to Engineers-Theory, Applications, and Software, *Journal of Manufacturing Systems*; 24, 3; ProQuest Research Library.

[2] Cotoia, M., Johnson, S., (2001). “Applying the axiomatic approach to business process redesign”, *Business Process Management Journal*, **7**,4, ABI/INFORM Global, 304

[3] Gonzales,I., dan Sanchez,I., 2010, Variable selection for Multivariate Statistical Process Control, *Journal of Quality Tehnology*, Juli 2010, Vol. 42, Iss 3, 242.

[4] Suh, N.P., 2001, *Axiomatic Design, Advances and Application*, New York, Oxford University Press.

[5] Triwidiastuti (2006), Disertasi, Data of service quality in air transportation.

AN APPLICATION OF ARIMA TECHNIQUE IN DETERMINING THE RAINFALL PREDICTION MODELS OVER SEVERAL REGIONS IN INDONESIA

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Abstract. In this present study, we mainly concerned an application of ARIMA (*Auto-Regressive Integrated Moving Average*) technique in determining the rainfall prediction models over several regions in Indonesia. They are Lampung (South Sumatera), Pandeglang (West Java), Indramayu (West Java), Banjarbaru (South Kalimantan), and Sumbawa Besar (Nusa Tenggara Barat). We used the monthly of surface rainfall data for period of January 1970 to December 2000. This study is motivated by the importance of understanding the mechanism of air and sea interaction between Monsoon and *El-Niño* event when they come simultaneously as already recommended by the IPCC (*Intergovernmental Panel on Climate Change* (*IPCC*) AR (*Assessment Report*) 4 and GEOSS (*Global Earth Observation System to System*). By applying the *Power Spectral Density* (PSD) and *Wavelet* analysis on those rainfall anomalies data, and also the Monsoon global index data, represented by the AUSMI (*Australian Monsoon Index*) and WNPMI (*Western North Pacific Monsoon Index*), we found a pre-dominant peak oscillation of them is about 12 month period that we call as the *Annual Oscillation* (AO). While, for the *El-Niño* event, represented by *SST* (*Sea Surface Temperature*) *Niño 3.4,* we found 60 month period. Furthermore analysis, we found more significant relationship between rainfall anomalies and AUSMI, WNPMI, and *SST Niño 3.4*. Please note here, this study was undertaken with assumption that Monsoon and *El-Niño* is interacted each other, and the model is developed by using multivariate regression method that formulated by Rainfall = a + b [AUSMI] + c [WNPMI] + d [SST Niño3.4]. By applying the ARIMA technique, we found the rainfall prediction models. They ARIMA $(1,0,1)^{12}$, with model equation $Z_t = 0.9989Z_{t-12} - 0.9338a_{t-12} + a_t$ (for AUSMI), ARIMA $(1,1,1)^{12}$, with model equation $Z_t = -0.0674Z_{t-12} + Z_{t-12} - Z_{t-24} - 0.9347a_{t-12} + a_t$, (for WNPMI), and ARIMA (2,0,2), with model equation $Z_t = 3.594Z_{t-1} - 0.8362Z_{t-2} - 1.634a_{t-1} - 0.1053a_{t-2} + a_t$ (for SST Niño 3.4). By applying these techniques, we can predict the rainfall behavior over those area until the end of 2013.

Keywords: ARIMA Technique and Rainfall Prediction Models

1. Introduction

As a national research institute with one of the basic tasks and functions it handles weather and climate issues, the *National Institute of Aeronautics and Space* (LAPAN), especially of the *Atmospheric Science and Technology Center* of LAPAN in Bandung which does have a vision and mission field of the climate change do everything possible to be able to contribute ideas or real contribution, particularly related to issues of climate predictions, especially predictions of rainfall anomalies that occurs in several regions in Indonesia.

There is one serious problem issues facing the Government (in this case the National Food Security Council) recently, ie, the impact of global climate change, especially the arrival of the dry season or wet season length increasingly difficult to predict properly, let alone on time and on target.

This problem arises because the climate prediction models, especially the model prediction of rainfall anomalies that exist today, are generally not fully consider the interconnection or teleconnections or interactions that occur between the various global climatic phenomena. There are only a few have already started applying it, as did the Harijono[1]. The prediction model developed at this time, generally still single column, individual, local, and rough resolution, but that sometimes happens interconnection mutually reinforcing, but sometimes mutual weaken.

Experience we have, when a drought occurs (more than six months) in 1997, followed by the long wet season (also more than six months) one year later, is the result of merging two natural phenomena, namely *El-Niño* and *Dipole Mode* in the same time period (known as simultaneous).

These events teach us to understand more deeply the mechanism of merging of the two phenomena in the well and correctly. If only the *El-Niño* or *La Niña* are coming, then it is not going to impact as severe if not followed by the presence of *Dipole Dipole Mode Mode Positive* or *Negative*.

The basic idea of this research is based on the national needs of the importance of monitoring the early indications (as *a precursor*) of impending extreme climate (especially rainfall extremes) in the western part of Indonesia region, which is relatively wet throughout the year. Then, prediction becomes very important, when the resulting model is not timely or not well targeted. Many models have been produced, based either dynamic or statistical. One of the statistical model that commontly used is ARIMA (*Auto-Regressive Integrated Moving Average*) or called as the *Box-Jenkins* method. However, there is one problem arises, when the model was no longer accurately applied to a particular region, although the data analysis used is long relatively.

Among the climatic parameters, rainfall is the most dominant elements of the climate in Indonesia. In general, rainfall in Indonesia is dominated by two types of Monsoon which is characterized by the rainy and dry season. Refer to the annual Monsoon cycle that set explicitly state of the atmosphere during the dry phase (dry) and wet phases (rain). This annual cycle split phase wet and dry phases into two periods.

Although Monsoons occur periodically, but the beginning of rainy season and the dry season is not always the same throughout the year. This is due to the season in Indonesia is influenced by several phenomena not only by Monsoon itself. They are the *El-Niño, Indian Ocean Dipole* (IOD), and also the local influence. If Monsoon related variations in annual precipitation, the *El-Niño* and IOD associated with variation in inter-annual rainfall.

Climate phenomenon affects the agricultural sector in Indonesia. The incident shows the increasingly important role with the emergence of extreme climatic conditions that has serious impacts on agricultural production such a shift in rainfall patterns and changes in surface air temperature (IPCC)[2]. Some concrete examples when the long dry season followed by a long wet season is from 1997 to 1998. At that period in Indonesia experienced a long dry season is almost 9 months old and is also a long wet season with nearly the same period.

Relationship between Monsoon and *El Niño* related needs to be studied more deeply their impact on global climate. Basically the process is gradual climatic aberrations. Therefore, an attempt to anticipate the effects of climate deviations should be understood as a whole start of the process until the proper handling and true impact. Therefore, it is necessary the development of science that combines the atmosphere and oceans, including interactions to account for the next several years.

The accuracy and reliable models that will greatly assist in the development of climate forecast information. However, research that examines the physical and dynamical interactions between the *El Niño* and Monsoon in Indonesia are still rare. This is presumably due to the complexity of the Monsoon dynamics in random parts of Indonesia affected by some type of Monsoon, like the Monsoon Asia, India, the Pacific, and Australia.

On this basis, the research was conducted with the primary objective to determine the effect of the interaction than the Monsoon and the *El Niño* when the two come together (*simultaneously*) to fluctuations in rainfall in some areas in Indonesia, as well as to determine the predictive model of rainfall in some areas.

2. A Brief Review of Box-Jenkins Models

The Box-Jenkins ARMA model is a combination of the AR and MA models. They are

$$\mathrm{X}_t = \delta + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} +$$
$$A_t - \theta_1 A_{t-1} - \theta_2 A_{t-2} - \ldots - \theta_q A_{t-q}$$

where the terms in the equation have the same meaning as given for the AR and MA model.

A couple of notes on this model.

1. The Box-Jenkins model assumes that the time series is stationary. Box and Jenkins recommend differencing non-stationary series one or more times to achieve stationarity. Doing so produces an ARIMA model, with the "I" standing for "Integrated".
2. Some formulations transform the series by subtracting the mean of the series from each data point. This yields a series with a mean of zero. Whether you need to do this or not is dependent on the software you use to estimate the model.
3. Box-Jenkins models can be extended to include seasonal autoregressive and seasonal moving average terms. Although this complicates the notation and mathematics of the model, the underlying concepts for seasonal autoregressive and seasonal moving average terms are similar to the non-seasonal autoregressive and moving average terms.

4. The most general Box-Jenkins model includes difference operators, autoregressive terms, moving average terms, seasonal difference operators, seasonal autoregressive terms, and seasonal moving average terms. As with modeling in general, however, only necessary terms should be included in the model. Those interested in the mathematical details can consult Box, Jenkins and Reisel[3], Chatfield[4], or Brockwell and Davis[6].

There are three primary stages in building a Box-Jenkins time series model. They are (1). Model Identification, (2). Model Estimation, and (3). Model Validation

Box-Jenkins Model Identification

The first step in developing a Box-Jenkins model is to determine if the series is stationary and if there is any significant seasonality that needs to be modeled. Stationarity can be assessed from a run sequence plot. The run sequence plot should show constant location and scale. It can also be detected from an autocorrelation plot. Specifically, non-stationarity is often indicated by an autocorrelation plot with very slow decay.

Seasonality (or periodicity) can usually be assessed from an autocorrelation plot, a seasonal subseries plot, or a spectral plot. Box and Jenkins recommend the differencing approach to achieve stationarity. However, fitting a curve and subtracting the fitted values from the original data can also be used in the context of Box-Jenkins models.

At the model identification stage, our goal is to detect seasonality, if it exists, and to identify the order for the seasonal autoregressive and seasonal moving average terms. For many series, the period is known and a single seasonality term is sufficient. For example, for monthly data we would typically include either a seasonal AR 12 term or a seasonal MA 12 term.

For Box-Jenkins models, we do not explicitly remove seasonality before fitting the model. Instead, we include the order of the seasonal terms in the model specification to the ARIMA estimation software. However, it may be helpful to apply a seasonal difference to the data and regenerate the autocorrelation and partial autocorrelation plots. This may help in the model idenfitication of the non-seasonal component of the model. In some cases, the seasonal differencing may remove most or all of the seasonality effect.

Once stationarity and seasonality have been addressed, the next step is to identify the order (i.e., the p and q) of the autoregressive and moving average terms. The primary tools for doing this are the autocorrelation plot and the partial autocorrelation plot. The sample autocorrelation plot and the sample partial autocorrelation plot are compared to the theoretical behavior of these plots when the order is known.

Specifically, for an AR(1) process, the sample autocorrelation function should have an exponentially decreasing appearance. However, higher-order AR processes are often a mixture of exponentially decreasing and damped sinusoidal components.

For higher-order autoregressive processes, the sample autocorrelation needs to be supplemented with a partial autocorrelation plot. The partial autocorrelation of an AR (p) process becomes zero at lag p+1 and greater, so we examine the sample partial autocorrelation function to see if there is evidence of a departure from zero. This is usually determined by placing a 95% confidence interval on the sample partial autocorrelation plot (most software programs that generate sample autocorrelation plots will also plot this confidence interval). If the software program does not generate the confidence band, it is approximately +/-2/SQRT(N), with N denoting the sample size.

The autocorrelation function of a MA(q) process becomes zero at lag q+1 and greater, so we examine the sample autocorrelation function to see where it essentially becomes zero. We do this by placing the 95% confidence interval for the sample autocorrelation function on the sample autocorrelation plot. Most software that can generate the autocorrelation plot can also generate this confidence interval.

The sample partial autocorrelation function is generally not helpful for identifying the order of the moving average process. The following table summarizes how we use the sample autocorrelation function for model identification.

In practice, the sample autocorrelation and partial autocorrelation functions are random variables and will not give the same picture as the theoretical functions. This makes the model identification more difficult. In particular, mixed models can be particularly difficult to identify.

Although experience is helpful, developing good models using these sample plots can involve much trial and error. For this reason, in recent years information-based criteria such as FPE (Final Prediction Error) and AIC (Aikake Information Criterion) and others have been preferred and used. These techniques can help automate the model identification process. These techniques require computer software to use. Fortunately, these techniques are available in many commerical statistical software programs that provide ARIMA modeling capabilities. For additional information on these techniques, see Brockwell and Davis[5,6].

Box-Jenkins Model Estimation

Estimating the parameters for the Box-Jenkins models is a quite complicated non-linear estimation problem. For this reason, the parameter estimation should be left to a high quality software program that fits Box-Jenkins models. Fortunately, many commerical statistical software programs now fit Box-Jenkins models.

The main approaches to fitting Box-Jenkins models are non-linear least squares and maximum likelihood estimation. Maximum likelihood estimation is generally the preferred technique. The likelihood equations for the full Box-Jenkins model are complicated and are not included here. See Brockwell and Davis[5,6] for the mathematical details. The Negiz case study shows an example of the Box-Jenkins model-fitting.

Box-Jenkins Model Validation

Model diagnostics for Box-Jenkins models is similar to model validation for non-linear least squares fitting. That is, the error term At is assumed to follow the assumptions for a stationary univariate process. The residuals should be white noise (or independent when their distributions are normal) drawings from a fixed distribution with a constant mean and variance. If the Box-Jenkins model is a good model for the data, the residuals should satisfy these assumptions.

If these assumptions are not satisfied, we need to fit a more appropriate model. That is, we go back to the model identification step and try to develop a better model. Hopefully the analysis of the residuals can provide some clues as to a more appropriate model.

As discussed in the EDA chapter, one way to assess if the residuals from the

Box-Jenkins model follow the assumptions is to generate a 4-plot of the residuals and an autocorrelation plot of the residuals. One could also look at the value of the Ljung-Box statistic[7].

An example of analyzing the residuals from a Box-Jenkins model is given in the Negiz data case study.

3. Data and Method of Analysis

The main data used in this study is the monthly of rainfall data at Lampung, Sumbawa Besar, Indramayu, Banjarbaru, and Pandeglang for period of 1976 to 2000 as described in Figure 1. The Sea Surface Temperature (SST) *Niño 3.4* data taken from (*http://www.cpc.noaa.gov/data/indices/ nino34.mth.ascii.txt*) for period of 1950 to 2009, the Monsoon Index data for the same period time, represented by the *Australian Monsoon Index* (AUSMI), and *Western North Pacific Monsoon Index* (WNPMI) taken from (*http://iprc.soest.hawaii.edu/users/ykaji/monsoon/realtime-monidx.html*).

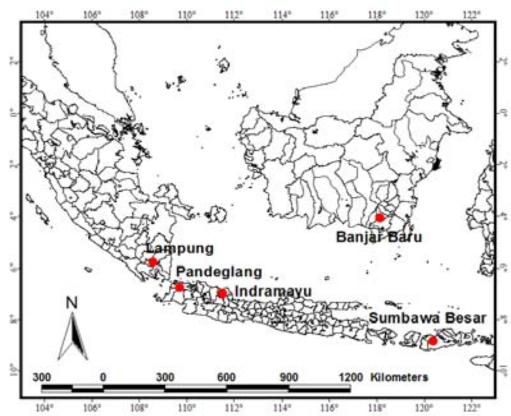


Figure 1: The five an investigated regions

While the method of analysis used is divided in two phases, each spectral analysis (FFT and WL) and statistical analysis (cross-correlation, multivariate, and Box-Jenkins approach). Special for Box-Jenkins approach, is divided into several stages, each model identification, parameter estimation models, testing or validation of the model, the determination of ARIMA models are relatively most suitable, and prediction 2013.

4. Results and Discussions

4.1. Multivariate Regression Analysis

We are showing firstly interconnection between Monsoon and *El-Niño* when they come simultaneously near real time as shown in Figure 2 below. We can see an a good pattern between rainfall anomalies behavior and all investigated parameters, represented by AUSMI, WNPMI, SST *Niño* 3.4, including their interconnections.

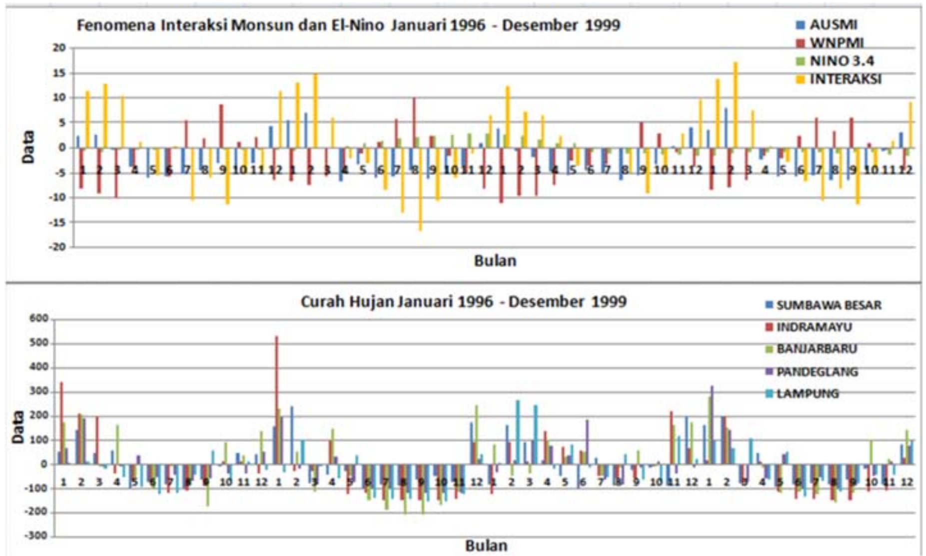


Figure 2: The time-series of Global Monsoon Index, SST *Niño* 3.4, and Rainfall anomalies for period of January 1996 to December 1999

By assuming the rainfall anomalies id depend on the global climate indices, represented by the AUSMI, WNPMI, and SST *Niño 3.4,* we can make a simple formula, that is (Hermawan, 2010)[5]:

R = f (AUSMI, WNPMI, SST *Nino 3.4*)

where R, and f is showing rainfall and function, respectively.

Fron this assumption, we got a multivariate regression as shown at Table 1 below. Please note, since the rainfall data is started from January 1976, we present here comparison between those data started from January 1976 to December 1999.

Table 1. The multivariate regressions between rainfall anomalies and AUSMI, WNPMI, and SST *Niño* 3.4 for period of January 1976 to December 1999

Location	CCF	Lag time (month)	STD	r^2	Multivariate Regression Formula
Sumbawa Besar	0.69	0	78.23313	0.532	Y = 16.267X1 - 3.940X2 - 6.790X3 + 6.819
Indramayu	0.64	0	124.86441	0.445	Y = 20.549X1 - 5.976X2 - 15.042X3 + 10.883
Banjarbaru	0.76	0	84.99061	0.591	Y = 8.195X1 - 12.388X2 - 25.964X3 + 7.408
Pandeglang	0.70	0	74.67462	0.535	Y = 15.923X1 - 3.445X2 - 5.264X3 + 6.509
Lampung	0.69	0	75.64186	0.486	Y = 9.954X1 - 6.721X2 - 2.913X3 + 6.593

Please note : X1=AUSMI, X2=WNPMI, and X3=SST *Niño* 3.4

Multiple regression equation above describes the respective roles of climate phenomena, the Monsoon and the *El-Niño* in influencing the behavior of rainfall in the study area. It is used to create a rainfall model that can eventually be used to predict rainfall in the study area. Relationship or correlation between rainfall and rainfall models with observations is explained by the size of the correlation coefficient (r^2) as described by Hasan[8].

Based on the analysis CCF (*Cross Correlation Function*), Banjarbaru have seen that the relative value of the CCF at most, approximately 0.76 compared to other regions. This suggests that Banjarbaru showed a relatively greater response time of the interconect various indices of global climate, especially when the phenomenon of Monsoon and the El-Niño united.

CCF analysis was also used to determine the time delay or time lag between phenomena Monsoon and *El-Niño* on rainfall. Positive sign (+) and negative (-) on the value of the CCF to orientate the relationship of two variables. If either has sign (+), meaning that the two variables have a proportional relationship, otherwise if the value of the CCF has a negative value (-), then the two variables are inversely linked.

Of Table 1 shows that the entire study area (Lampung, Sumbawa Besar, Indramayu, Banjarbaru and Pandeglang) has a value of CCF is positive (+). This suggests that the phenomenon of interaction between the Monsoon and the *El-Niño* on rainfall has a directly proportional relationship. That means they strengthened the interaction phenomenon when rainfall in the study area will increase, and vice versa weakening the interaction phenomena of rainfall in the study area will decrease.

In addition to knowing the value of the CCF, in Table 1dapat seen also how long lag time or delay time in some areas of study. Lag time or delay time is the time taken by the phenomenon of interaction of Monsoon and *El-Niño* to affect rainfall in the study area. In Table 1 above it can be seen that the entire study area (Lampung, Sumbawa Besar, Indramayu, Banjarbaru, and Pandeglang) has a time lag of 0 months. It means that the interaction between the incident and the *El-Niño* Monsun has no delay time to influence rainfall in the region with the original data.

4.2. Verification of the Multivariate Regression

In the Figure 2 it can be seen that the regions of the five studies had a large correlation value. This suggests that the model is good and can be used to explain the rainfall events in the study area that have been affected by the interaction between the Monsoon with SST Niño 3.4. Validation of multivariate model of rainfall period January 2000 - December 2000 one year after 1999 as stated in Figure 3 below. Please note here, we for all validation or verification results, we got almost a good value, it’s almost 0.9 for all an investigated regions.

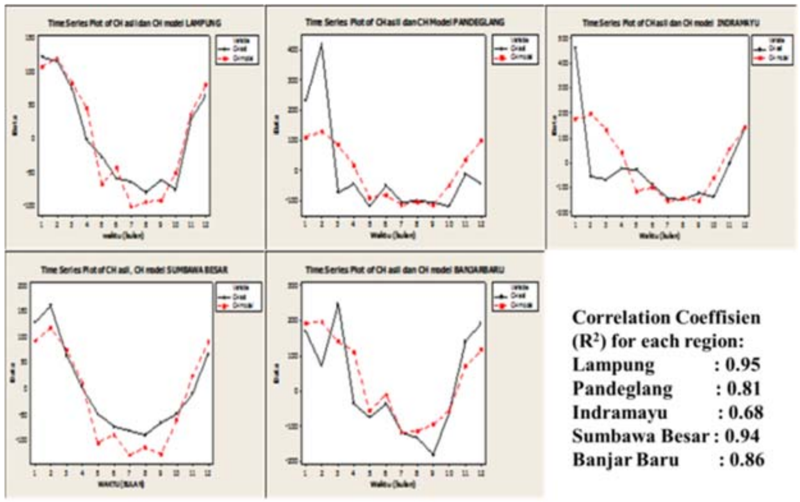


Figure 3. Validation of multivariate model of rainfall period January to December 2000.

4.3. Determining of the ARIMA Model Prediction

Before determined the ARIMA-based prediction model, the first step that needs to be done is to test the stationarity of data, they are AUSMI, WNPMI, and SST *Niño* 3.4. Afterwards, the identification and assessment models, testing or validation of the model, before finally forecasting or prediction.

4.3.1. Test of the Stationarity Data

Tests performed on the data stationary rainfall anomalies period January 1949 to December 1997 as calibration results of AUSMI, WNPMI, and SST *Niño* 3.4 with rainfall data for period of January 1976 to December 1999. This is done because it is a requirement modeling of time series data, because the data are not stationary hard to predict which cause the resulting model is not maximal. It should be noted that most of the time series data are nonstationary and aspects of AR (*Auto-Regressive*) and MA (*Moving Average*) of ARIMA models only pleased with the time series data are stationary.

Stationary data means there is no growth or decline in the data. Data seen by naked eye along the horizontal time axis. In other words, the data are fluctuations around a mean value that is constant, not depending on the time and the variance and thus appears to remain constant at all times.

To check stationarity can be viewed via the autocorrelation function, partial autocorrelation function, and plot time series of data to be examined stasioneritasnya. The data are not stationary data must be converted into stationary by differencing, ie, calculate the change or difference in value of observation. Obtained difference value, double-checked whether the data is stationary or not. If such data are not stationary then performed again differencing. If the variance is not stationary, then the logarithmic transformation.

Figure 4 shows an examination of the data for stationarity. The results of the examination showed that the data used are not stationary. Therefore the data is processed again by differencing technique, just one time differencing. After differencing seen that the data become more stationary and predictable than before differencing. ACF (*Auto Correlation Function*) and PACF (*Partial Auto Correlation Function*) in the entire study area has a seasonal pattern of rainfall so that the data is the data on the seasonal lag 1.

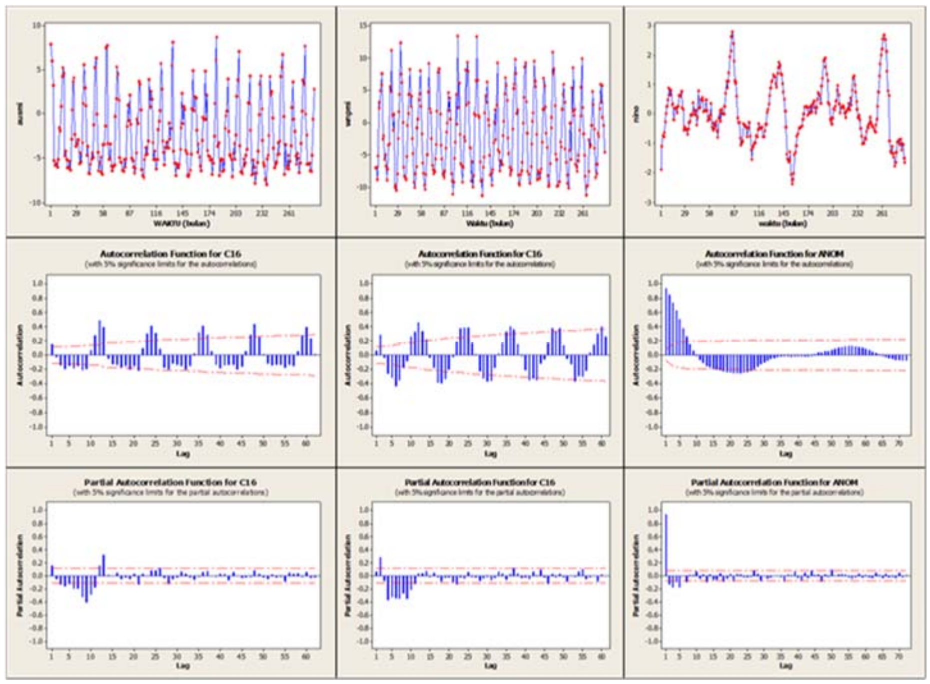


Figure 4. Plotting data, differencing, ACF, PACF for earch parameters, AUSMI, WNPMI, dan SST *Niño* 3.4 for period of January to 1976 to December 1999.

4.3.2. Identification and Estimated Model

Through the process of identification and assessment, while the obtained models of global climate index data plot is ARIMA $(1,0,1)^{12}$, ARIMA $(0,1,1)^{12}$, ARIMA $(1,1,1)^{12}$, and ARIMA (2,0,2). Obtained from all models while a suitable model for global climate data is ARIMA $(1,0,1)^{12}$ for AUSMI, ARIMA $(1,1,1)^{12}$ for WNMPI, and ARIMA (2,0,2) for SST *Niño* 3.4. ARIMA model on top of an equation obtained at each global climate (Table 2) where Z_t is the data at month t and at an error in month-t. Equation is then validated with observational data..

Table 2. Determining of the ARIMA Prediction Model

Parameter	ARIMA Formula	R^2
AUSMI $(1,0,1)^{12}$	$Z_t = 0.9989Z_{t-12} - 0.9338a_{t-12} + a_t$	0.94
WNPMI $(1,1,1)^{12}$	$Z_t = -0.0674Z_{t-12} + Z_{t-12} - Z_{t-24} - 0.9347a_{t-12} + a_t$	0.94
Nino 3.4 (2,0,2)	$Z_t = 3.594Z_{t-1} - 0.8362Z_{t-2} - 1.634a_{t-1} - 0.1053a_{t-2} + a_t$	0.95

4.3.3. Testing or Validation Model

Testing or validation is done by comparing the results of model calculations with observational data with the time period used to validate the model is January 2000 to December 2000 (Figure 5). Plot the data for the entire data ikim ARIMA models globally have a large correlation value is above 0.9. This shows that the ARIMA model is very good and can be used to explain events with El-NNO Monsun. Based on the ARIMA model equations get global climate time series data prediction in January 2013hingga December 2013 (Figure 5). The data is then developed again with multivariate equations, to obtain predictions of rainfall for the month of January 2013 - December 2013.

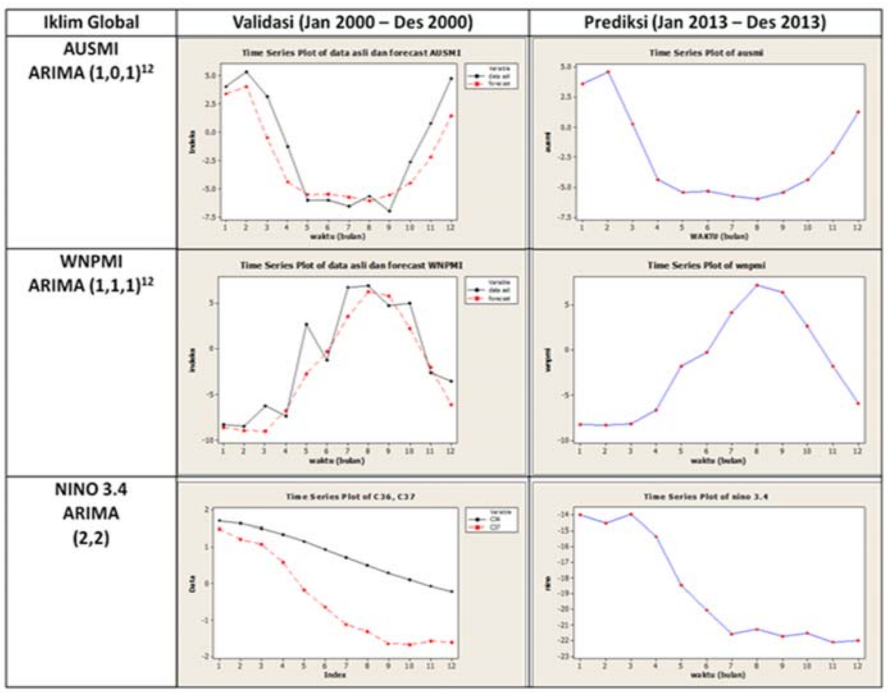


Figure 5. Plotting of validation and prediction of AUSMI, WNPMI, and SST *Niño* 3.4 for period of January to December 2000 (for validation), and January 2013 to December 2013, respectively.

4.3.4. Prediction Rainfall Anomalies Based on the ARIMA Model

After passing through the stages of ARIMA, the predictive value of the data obtained global climate phenomena (AUSMI, WNMPI, and SST Niño 3.4) in the period January 2013 to December 2013. Global climate data are then substituted into the multivariate equation in Table 1. Data obtained from the substitution of rainfall anomalies for each area of study period January 2013 - December 2013 (Figure 6). Thus, it can be interpreted that the multivariate equation can be used to explain the rainfall events that have been affected by the interaction that occurs between the *El-Niño* Monsoon.

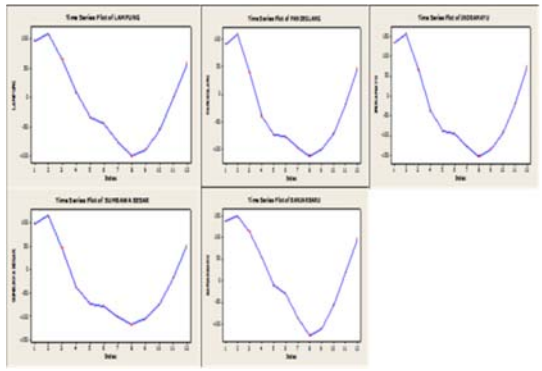


Figure 6. Prediction of rainfall anomalies over Lampung, Pandeglang, Indramayu, Sumbawa Besar, and Banjarbaru respectively for period of January to December 2013.

By looking at Figure 6 above, we can see the minimum peak of rainfall anomalies, mostly occur in around July. The pattern of those rainfall looks similar each other. It indicates that most region is mostly effected by the Monsoon system. We have another phenomena that called as *El-Niño,* but we suspect until the end of this year, the Monsoon system is still as the pre-dominant peak oscillation.

To get more information about our prediction, we present here the rainfall pattern over those area taken from the *in-situ* and satellite data observation as shown in Figure 7 below.

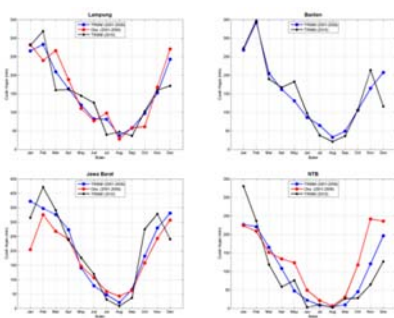


Figure 7. The rainfall pattern over Lampung, Pandeglang (Banten), Indramayu (West Java), and Sumbawa Besar (NTB)

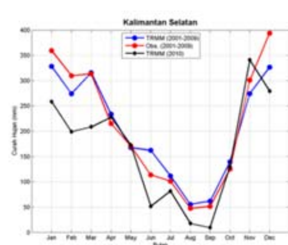


Figure 8. As the as Figure 7, but for Banjarbaru (South Kalimantan)

In the end our analysis, we are showing here, another technique that we call as the *composite technique analysis* that very useful for describing the rainfall behaviour over those area more than 30 years observation that commontly is mentioned as climatologist. We start analysis from Lampung, then following by Pandeglang, Indramayu, Sumbawa Besar, and Banjarbaru, respectively as shown in Figure 9 below.

- **Lampung**

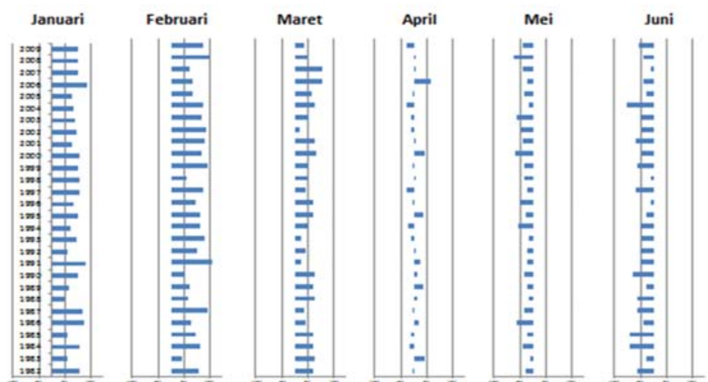


Figure 9(a). The rainfall pattern behaviour over Lampung from January to June for period of January 1982 to December 2009.

- **Lampung**

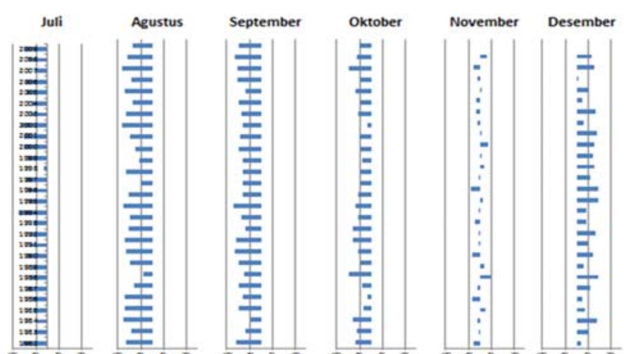


Figure 9(b). As the as Figure 9(a), but from July to December.

• **Pandeglang**

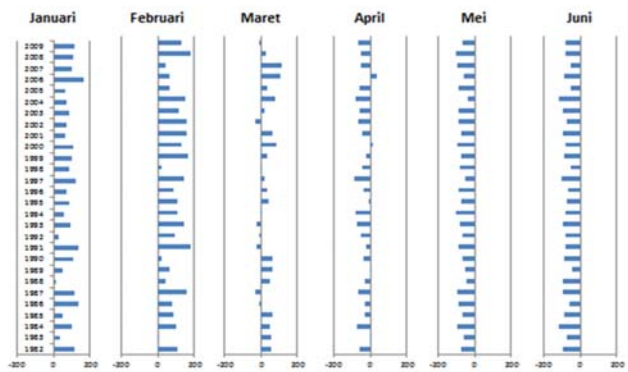


Figure 10(a). The rainfall pattern behaviour over Pandeglang from January to June for period of January 1982 to December 2009.

• **Pandeglang**

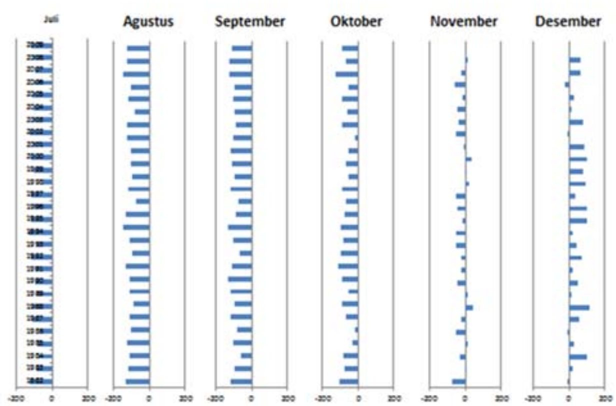


Figure 10(b). As the as Figure 10(a), but from July to December.

• **Indramayu**

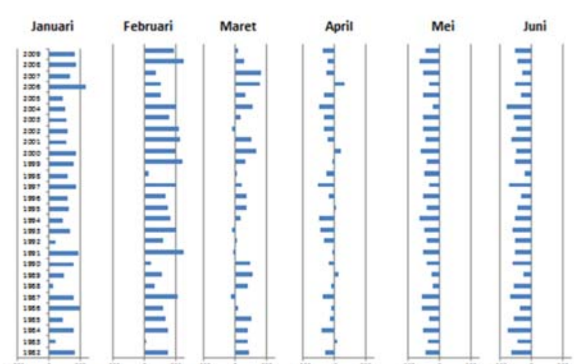


Figure 11(a). The rainfall pattern behaviour over Indramayu from January to June for period of January 1982 to December 2009.

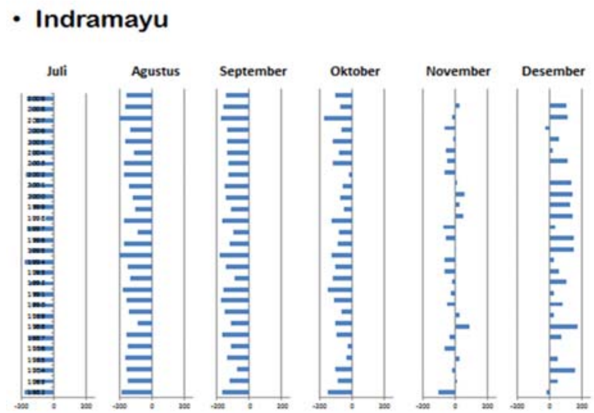


Figure 11(b). As the as Figure 11(a), but from July to December.

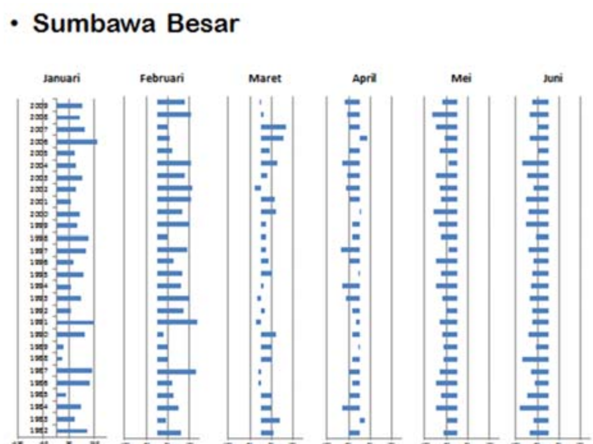


Figure 12(a). The rainfall pattern behaviour over Sumbawa Besar from January to June for period of January 1982 to December 2009.

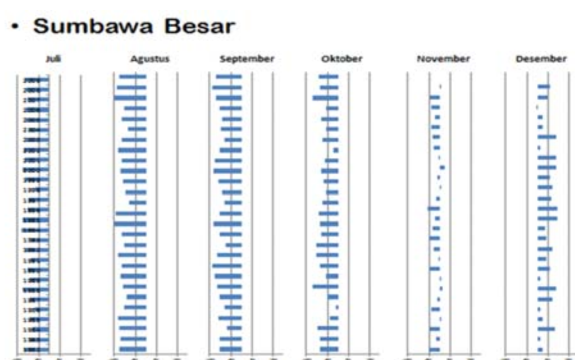
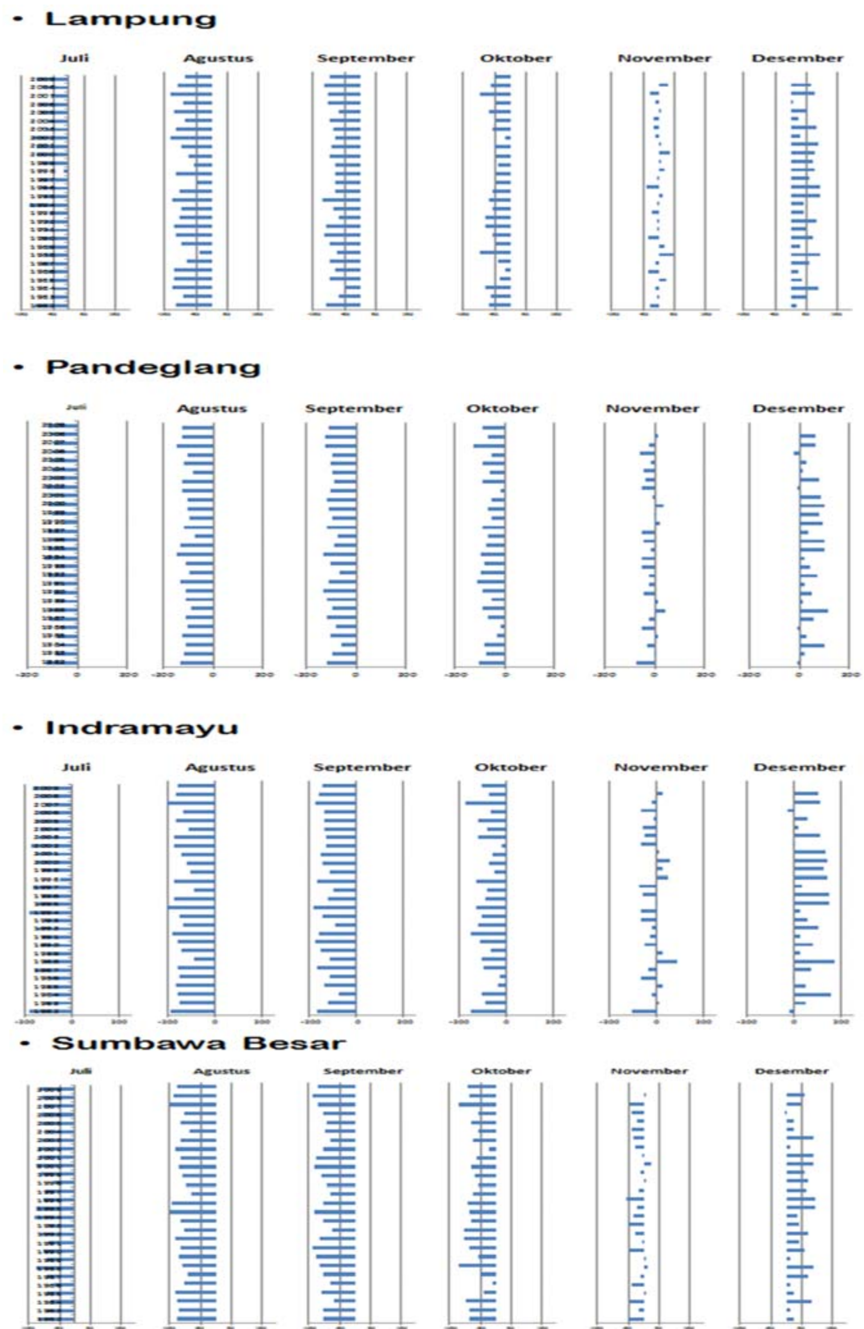


Figure 12(b). As the as Figure 12(a), but from July to December.

Since we are interest to predict the rainfall behaviour over those investigated area, we try to focus on investigating in November and December, respectively as shown in Figure 13 below.



- **Banjarbaru**

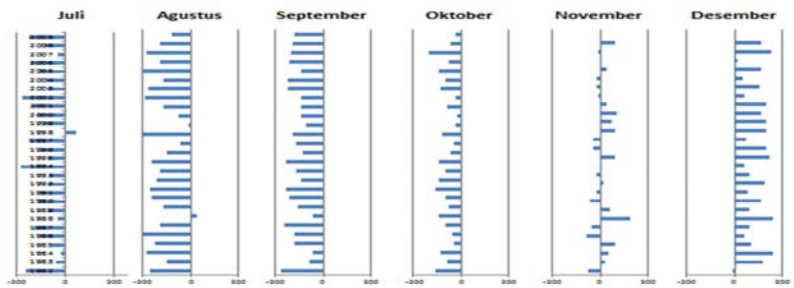


Figure 13. The rainfall pattern behaviour over Lampung, Pandeglang, Indramayu, Sumbawa Besar, and banjarbaru from July to December for period of January 1982 to December 2009.

Looking at the Figure 11 carefully, we obtain the general information that almost those area have the same rainfall pattern anomalies. We can see clearly that transition season from dry to wet season is occurred in November. While, the wet season will be started in December. We can suspect also that almost no extreme rainfall will be occurred, at least until the end of this year. They are almost the normal condition. This is will be occurred perfectly by assuming that *El-Niño* event still in normal condition, although we need to always investigate the behaviour of the Monsoon system, especially the AUSMI parameter that we known before this parameter has a strong correlation with the rainfall behaviour over Indonesia, including this our investigated area.

5. Concluding Remarks

Based on the analysis of the model using the Box-Jenkins method and through the process of identification, assessment and testing, the obtained prediction model ARIMA $(1,0,1)^{12}$ for AUSMI, ARIMA $(1,1,1)^{12}$ for WNPMI, and ARIMA (2,0,2) for SST *Niño* 3.4. The ARIMA equation for three indexes is $Z_t = 0.9989Z_{t-12} - 0.9338a_{t-12} + a_t$ (AUSMI), $Z_t = -0.0674Z_{t-12} + Z_{t-12} - Z_{t-24} - 0.9347a_{t-12} + a_t$, (WNPMI), and $Z_t = 3.594Z_{t-1} - 0.8362Z_{t-2} - 1.634a_{t-1} - 0.1053a_{t-2} + a_t$ (SST *Niño* 3.4). The ARIMA equation shows, for forecasting future depends on the data to the previous month and a-t-t error to the previous month.

The model obtained is the result of multiple regression analysis between the data anomalies of monthly rainfall in Lampung, Pandeglang, Indramayu, Sumbawa Besar, and Banjarbaru with global monsoon index data and also the data Niño 3.4 SST for the period January 1976 to December 1999. As for the model validation or test performed using the data one year later, in the period January to December 2000 which showed a good correlation between the data ARIMA models with original dat with a correlation coefficient close to 0.9. This shows that the ARIMA models developed can be used to predict rainfall anomalies throughout the study area.

Results predicted by the end of 2013 showed that the study area has the fifth general rainfall patterns the same approach, ie dominated by monsoonal pattern AUSMI parameters. Peak of the rainy season generally occurs during DJF (December-January-February), while the peak of the dry season, generally occurs during JAS (July-August-September). This happens in almost every area of study are examined.

Specially for the month of November and December 2013, the beginning of a season of transition or the transition from the dry season to the rainy season. In addition to the results of the analysis indicated that diapatj ARIMA model, also evidenced by the data of rainfall anomalies in each month using Hovmoller technique, which clearly visible November is a month of transition. While December is the beginning of the dry season. This prediction will run perfectly well and, of course, assuming that factors Niño 3.4 SST was normal until the end of December 2013. Likewise Monsoon index global parameters, especially AUSMI is in normal position..

For further research, to obtain an accurate prediction model using time series data should be longer and more complete. Moreover, to explain the distribution of rainfall areas with better spatial analysis using Hovmoller analysis. Future studies are expected to also be able to create a statistical model using other statistical methods such as CCA (Canonical Correlation Analysis).

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References

[1] Harijono, S.W.B. 2008. Analisis Dinamika Atmosfer di Bagian Utara Ekuator Sumatera Pada Saat Peristiwa *El-Niño* dan *Dipole Mode Positif* Terjadi Bersamaan, *Jurnal Sains Dirgantara* (JSD), 5(2), 130 – 148.

[2] IPCC. 2001 .Climate Change 2001 : Impact, Adaptation and Vulnerabi1ity. Cambridge: Cambridge University Press.

[3] Box, G. E. P., Jenkins, G. M., and Reinsel, G. C. (1994). Time Series Analysis, Forecasting and Control, 3rd ed. Prentice Hall, Englewood Clifs, NJ.

[4] Chatfield, C. (1996). The Analysis of Time Series, 5th ed., Chapman & Hall, New York, NY.

[5] Brockwell, Peter J. and Davis, Richard A. (1987). Time Series: Theory and Methods, Springer-Verlang..

[6] Brockwell, Peter J. and Davis, Richard A. (2002). Introduction to Time Series and Forecasting, 2nd. ed., Springer-Verlang.

[7] Ljung, G. and Box, G. (1978). "On a Measure of Lack of Fit in Time Series Models", Biometrika, 65, 297-303.

[8] Hasan, M. Iqbal. 2003. Pokok-Pokok Materi Statistik 2 (Statistik Iterensif) Edisi Kedua. Jakarta: Bumi Aksara.

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